Applied Compositional Thinking for Engineers

Session 7

Life is hard

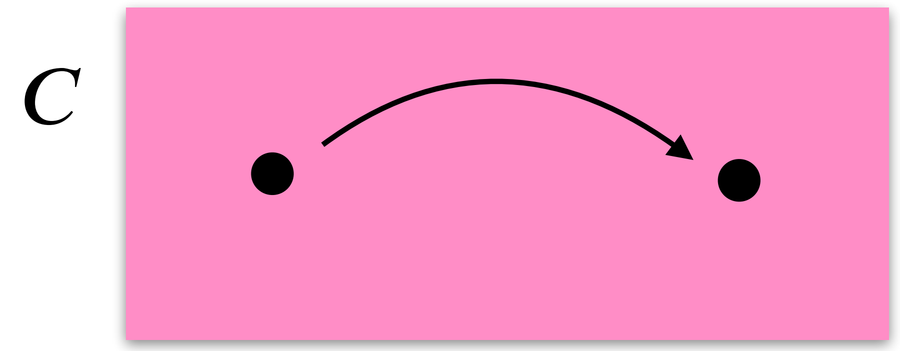
The story so far

- We have defined **what is a category** and provided several examples.
- We have defined notions of **product**, **coproduct**, **subcategory**, etc.
- We have been looking at **posets** and how they represent **trade-offs**.
- We have seen how **posets can be interpreted as categories**.

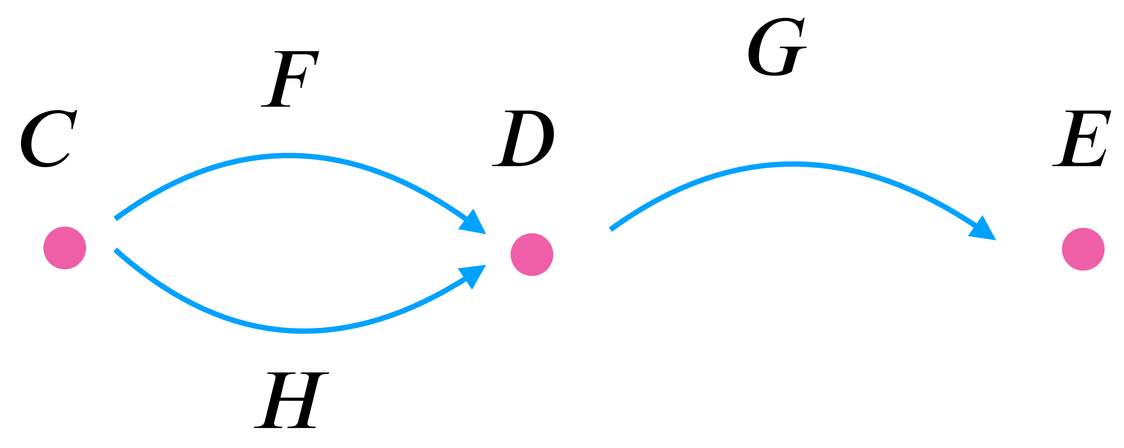
- Today:
  - **Monotone functions**
  - **Lattices;** meet, join. Interpretation as product/coproduct.
  - **Functors** as generalization of monotone functions on posets.

Looking ahead

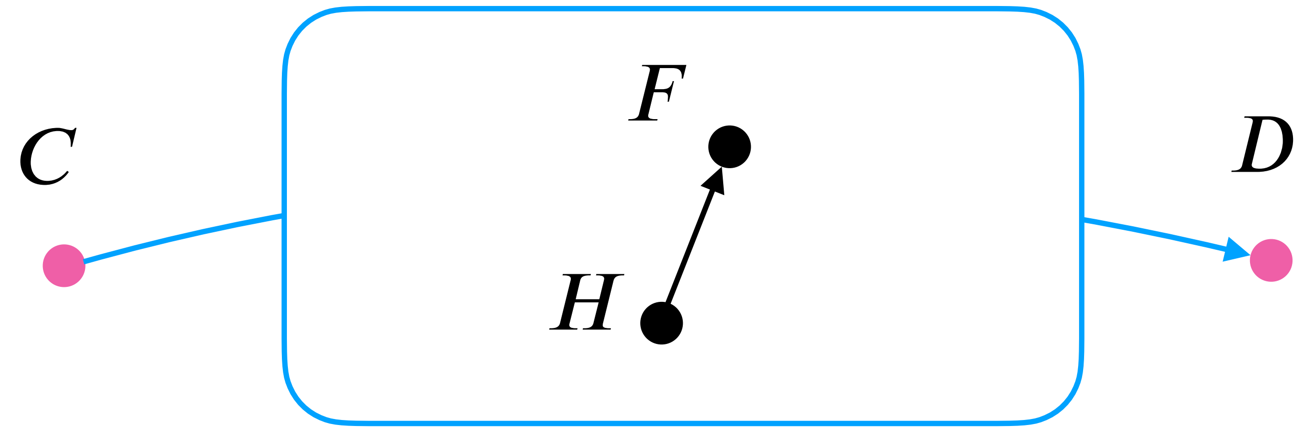
- In a category, morphisms are arrows between objects.



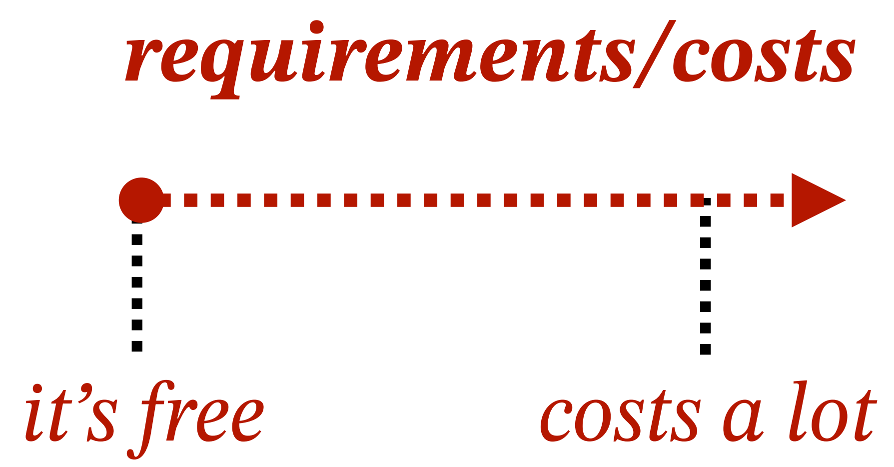
- In the **category of categories Cat**, **functors** are **arrows between categories**.

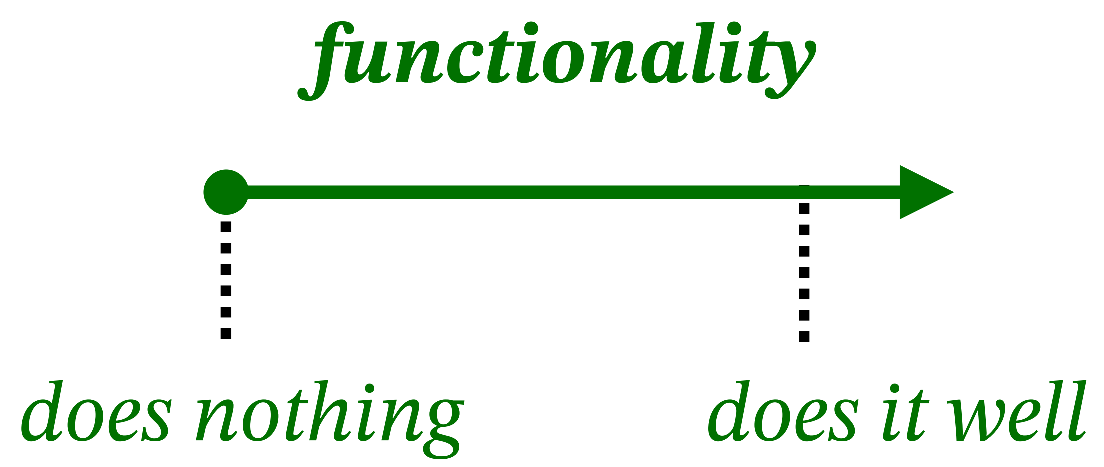


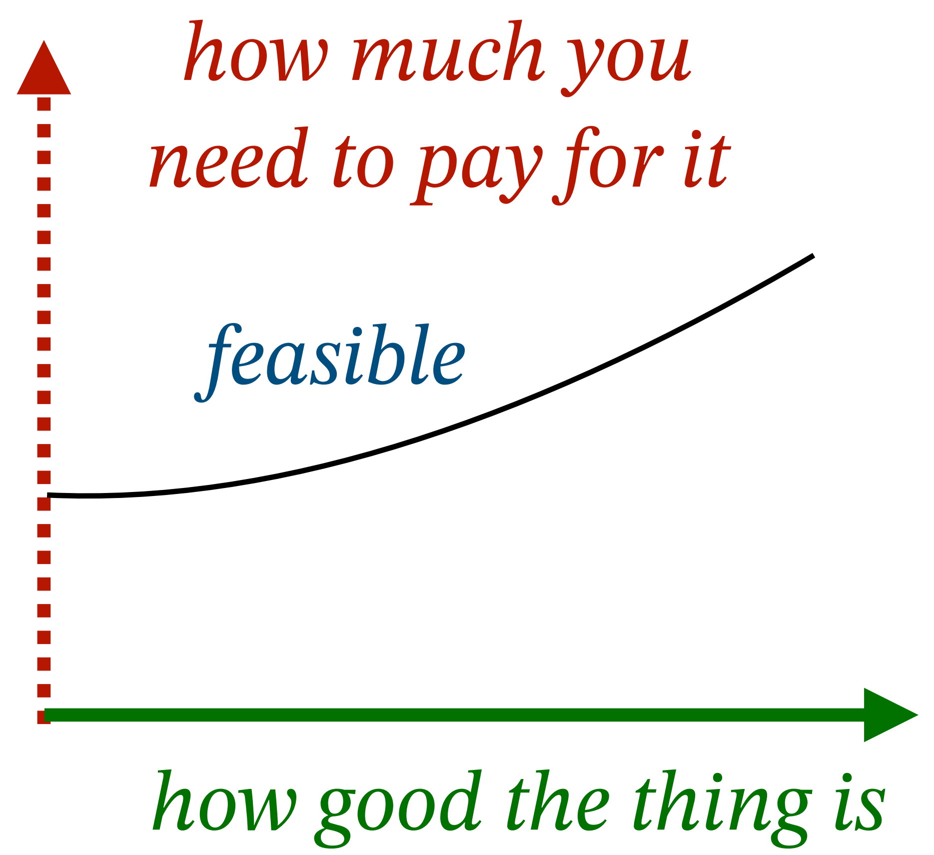
- **Natural transformations** are arrows between functors with same domain/codomain.

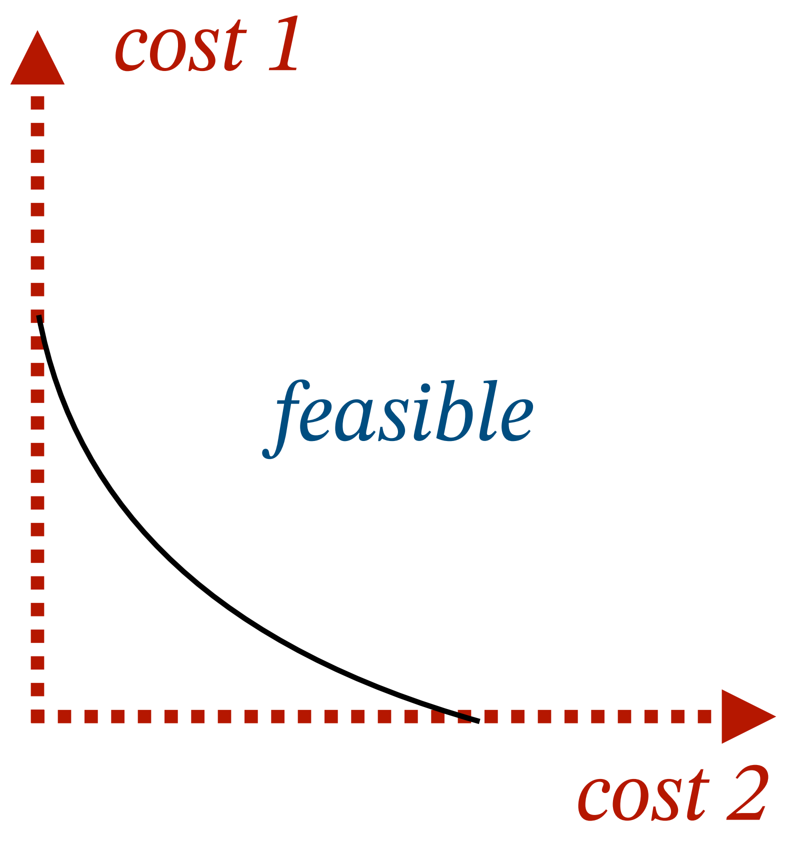


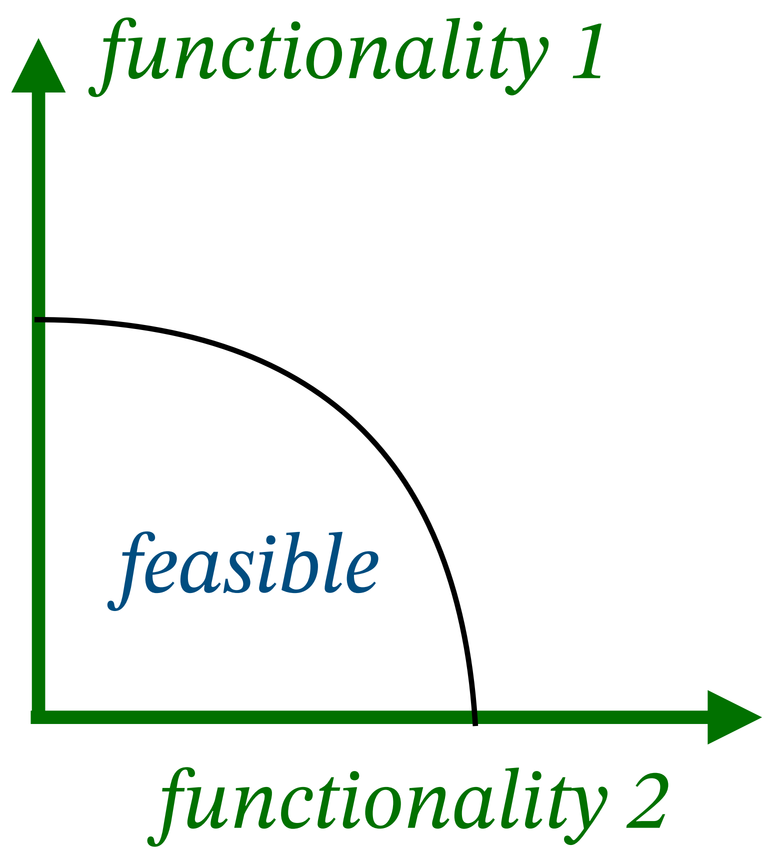
$\mathrm{Hom}_{\mathbf{Cat}}(C; D)$


requirements/costs
it's free
costs a lot


functionality
does nothing
does it well


how much you
need to pay for it
feasible
how good the thing is

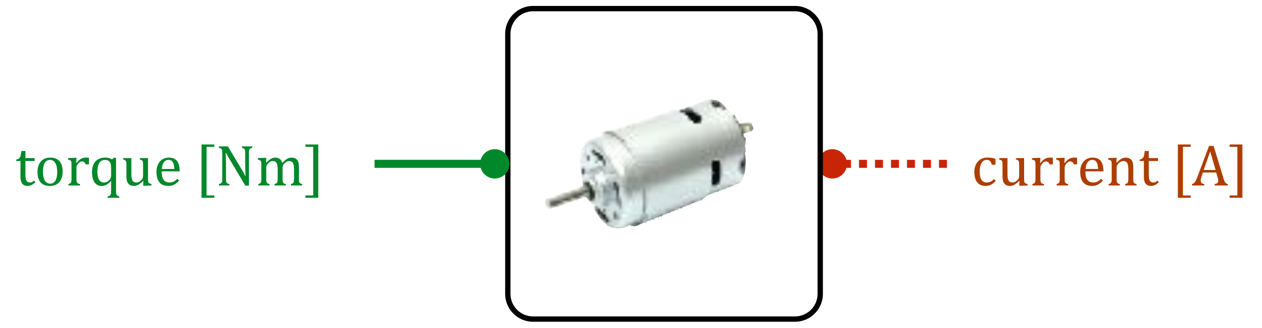

cost 1
feasible
cost 2


functionality 1
feasible
functionality 2

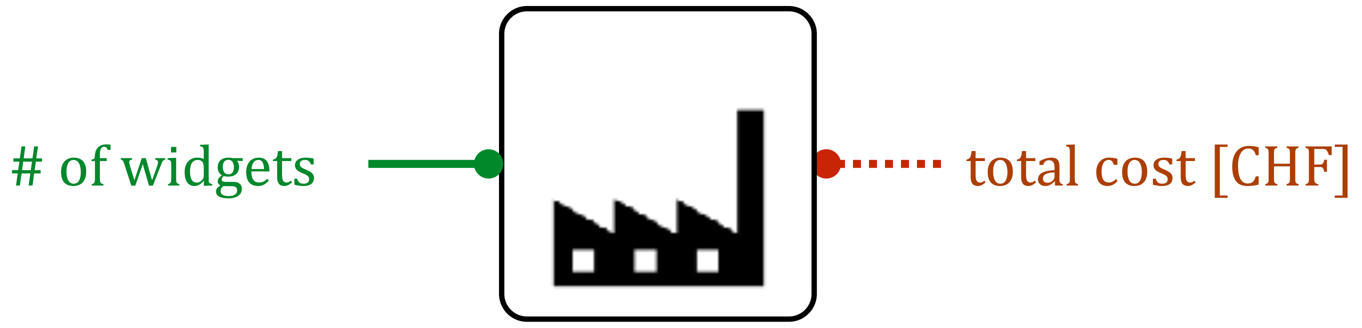
Monotone functions

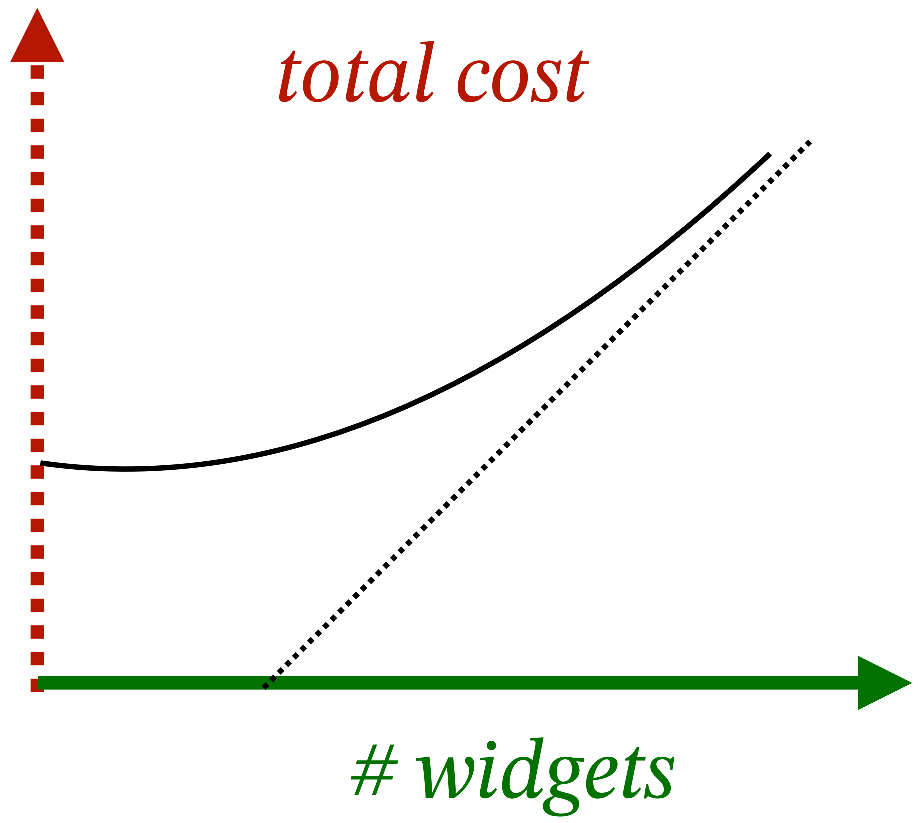
- Monotone functions on $\mathbb{R}$ = "**non-decreasing** functions".

- Examples: all dimensioning relations in engineering:



- Example: Manufacturing cost as a function of # of widgets.

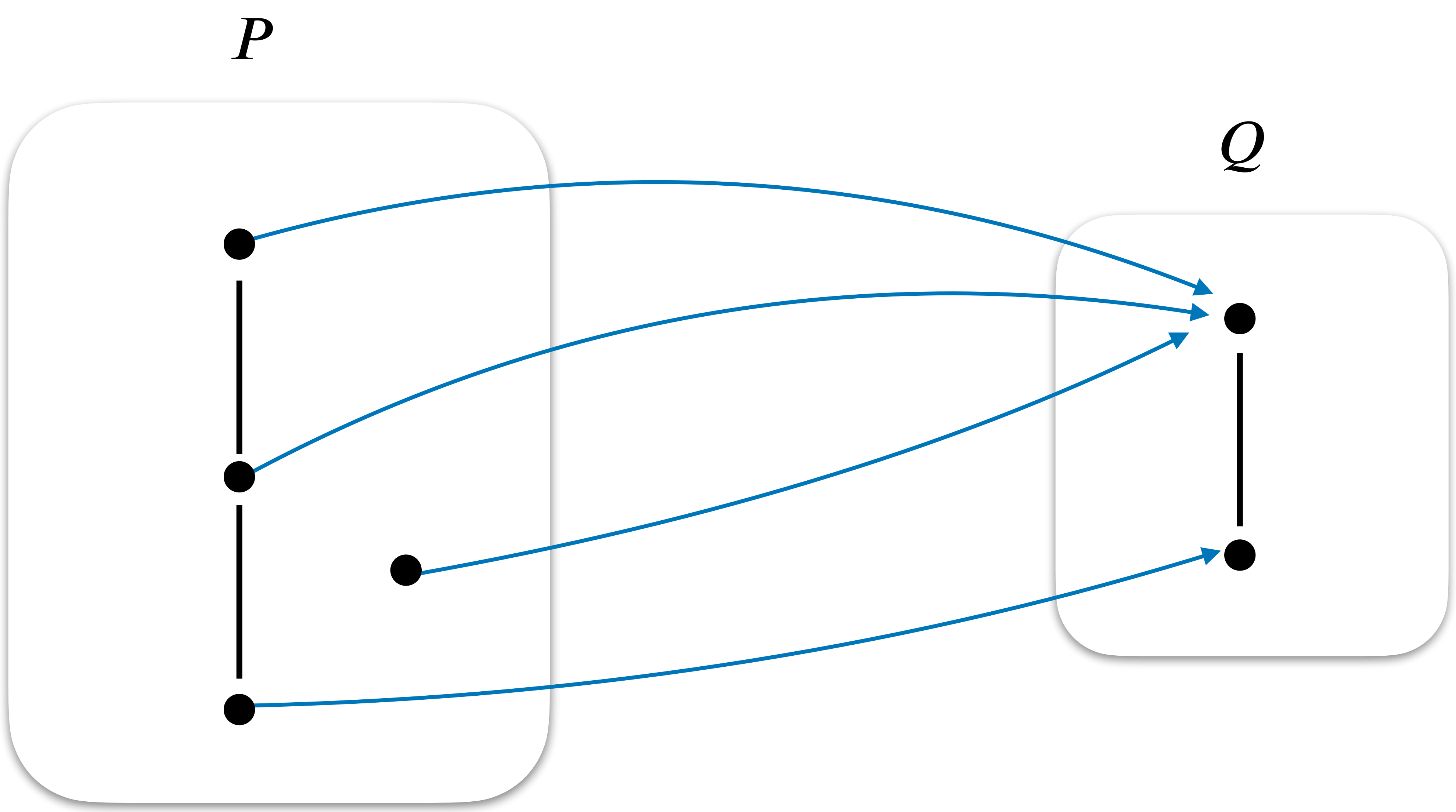




Monotone functions on posets

- A **function** $f : \langle P, \leq_P \rangle \to \langle Q, \leq_Q \rangle$ is **monotone** iff

$$\frac{a \leq_P b}{f(a) \leq_Q f(b)}$$



- It is an **order isomorphism** if it is **injective** and

$$\frac{a \leq_P b}{\overline{f(a) \leq_Q f(b)}}$$

Rounding functions

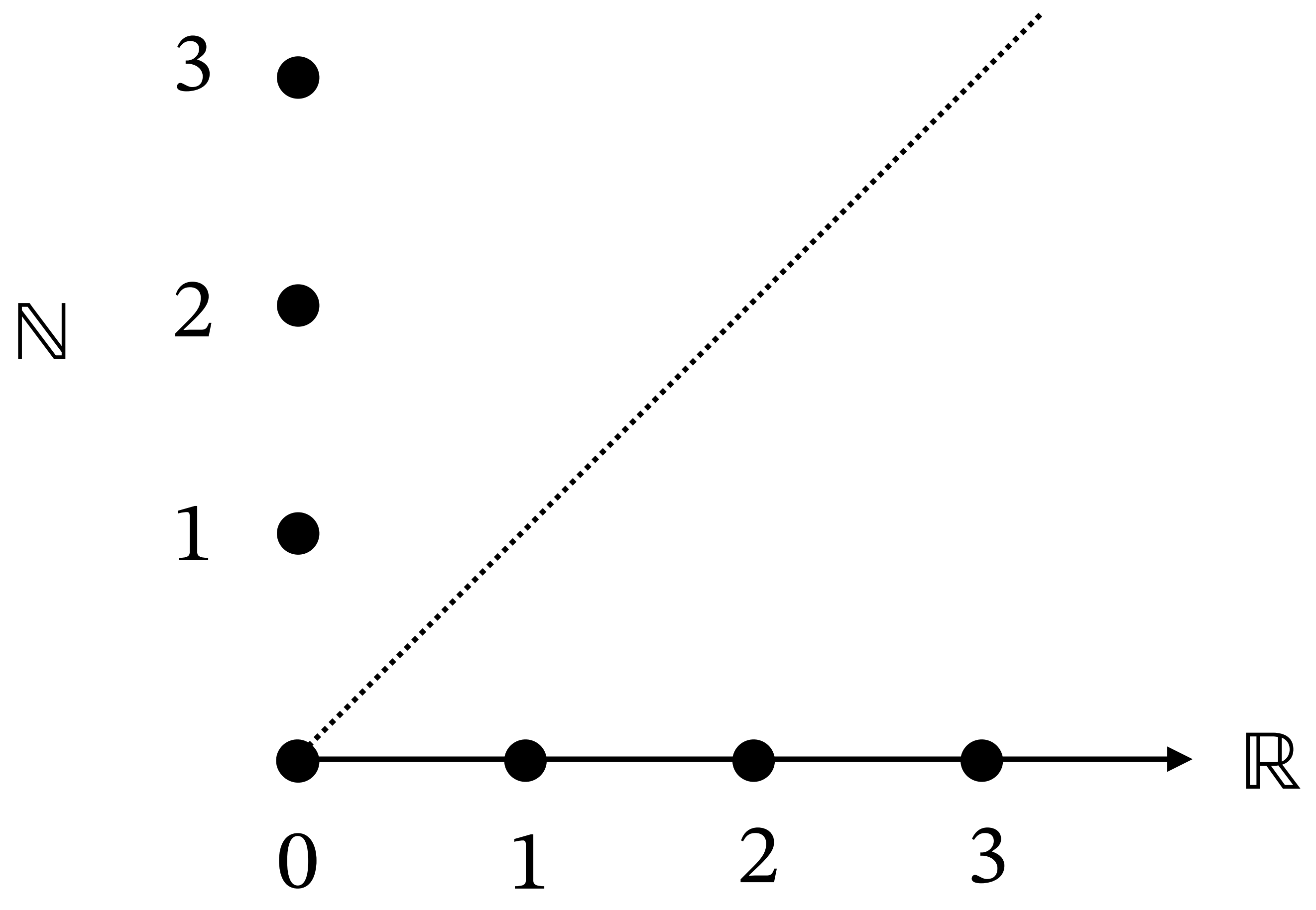
- There are a few **rounding functions**:

$$\mathsf{ceil} : \langle \mathbb{R}, \leq \rangle \to \langle \mathbb{N}, \leq \rangle$$

$$\mathsf{floor} : \langle \mathbb{R}, \leq \rangle \to \langle \mathbb{N}, \leq \rangle$$

$$\mathsf{rtntte} : \langle \mathbb{R}, \leq \rangle \to \langle \mathbb{N}, \leq \rangle$$

"round to nearest, ties to even"
(default IEEE-754)



- Also note:

$$\forall x \in \mathbb{R} : \quad \mathsf{floor}(x) \leq \mathsf{rtntte}(x) \leq \mathsf{ceil}(x)$$

$$\begin{tikzcd}[column sep=huge]
\langle \mathbb{R}, \leq \rangle \arrow[r, bend left=60, "\mathsf{ceil}"] \arrow[r, "\mathsf{rnttee}"] \arrow[r, bend right=60, "\mathsf{floor}"'] & \langle \mathbb{N}, \leq \rangle
\end{tikzcd}$$

Upper and lower bounds

- The **upper bounds** of a subset S of a poset P are, <u>if they exist</u>, the elements that dominate all elements in S.

- The **least upper bound**, <u>if it exists</u>, is the least among the upper bounds of S. We also call it the *join*, *supremum,* $\vee$ S.
- The least upper bound need not exist.

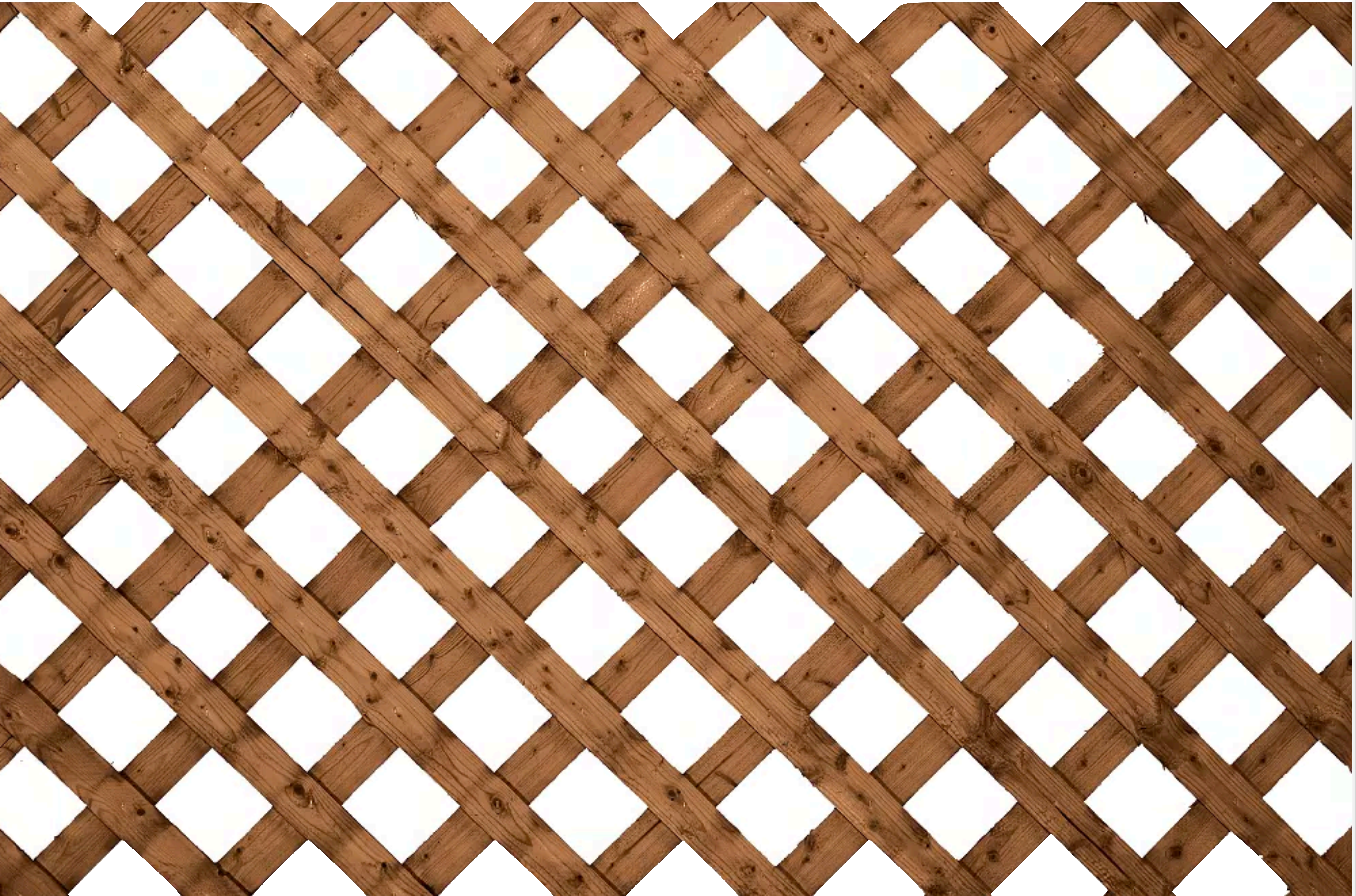
- The least upper bound need not exist even in a total order.

$$\mathbb{R}_{>0} = \{x \in \mathbb{R} \,:\, x > 0\}$$

- Dually: define **lower bound,** also called **meet, minimum,** $\wedge S$**.**

Lattices

- **Lattices** are posets where the **meet** $\wedge$ and the **join** $\vee$ always exist.



Lattices

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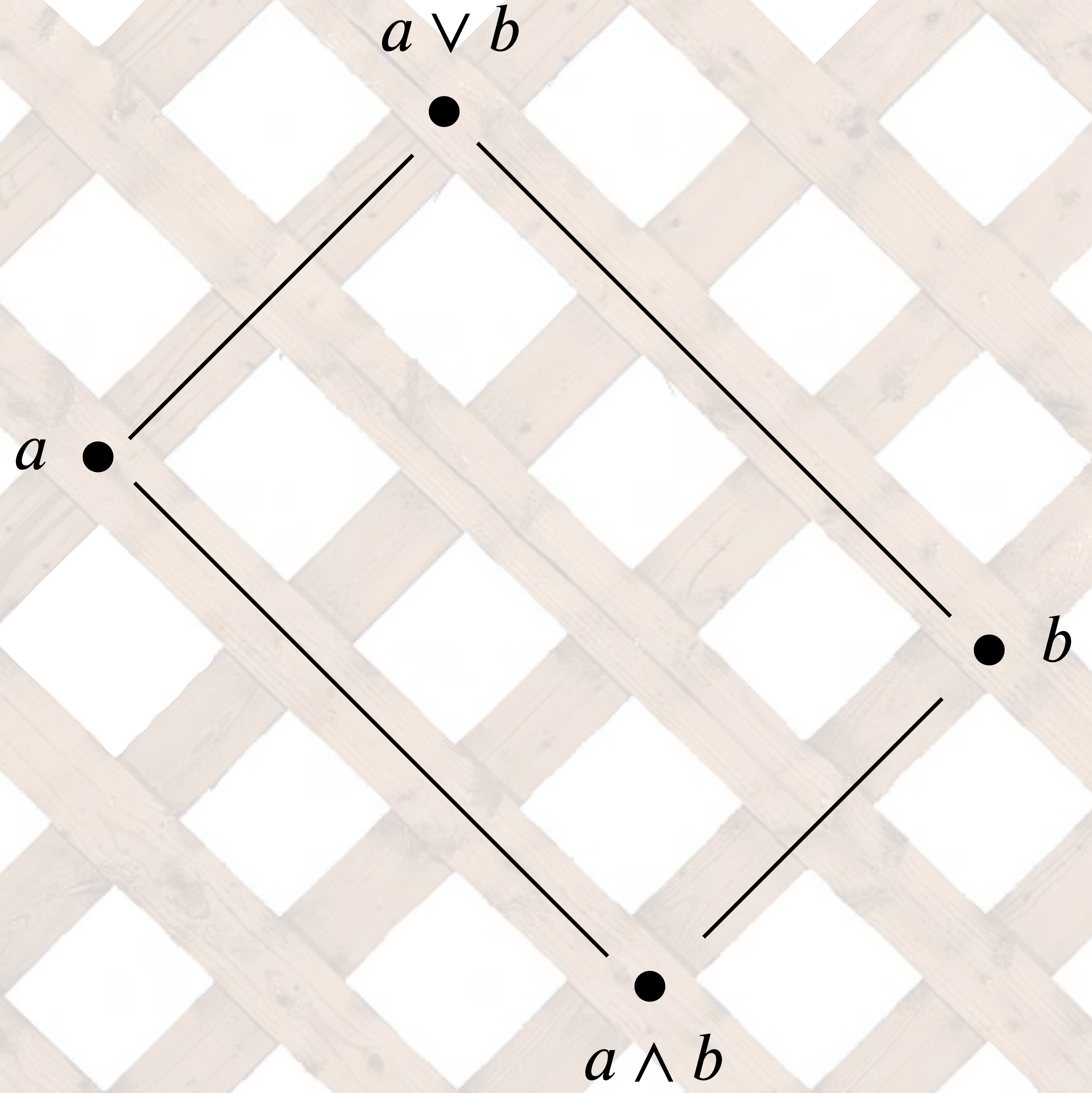


$$(a \wedge b) \ \leq \ a \ \leq \ (a \vee b)$$

Lattices

- **Lattices** are posets where the **meet** $\wedge$ and the **join** $\vee$ always exist.

- Example: Consider a poset **Propositions** where:
  - the elements are propositions (equivalence classes of propositions)
  - the partial order is given by

$$\frac{a \leq b}{a \Rightarrow b}$$

  - This is a lattice with the logical operations $\wedge$ and $\vee$.

$$\begin{tikzcd}
 & & \top & & \\
 & & a \vee b \vee c \arrow[u, Rightarrow] & & \\
 & a \vee b \arrow[ur, Rightarrow] & & b \vee c \arrow[ul, Rightarrow] & \\
a \arrow[ur, Rightarrow] & & b \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & c \arrow[ul, Rightarrow] \\
 & a \wedge b \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & b \wedge c \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \\
 & & a \wedge b \wedge c \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & \\
 & & \bot \arrow[u, Rightarrow] & &
\end{tikzcd}$$

$$\frac{a \Leftrightarrow b}{a = b}$$

$$\frac{\top}{a \Rightarrow a}$$

$$\frac{(a \Rightarrow b) \wedge (b \Rightarrow c)}{a \Rightarrow c}$$

Powersets

- **Lattices** are posets where the **meet** $\wedge$ and the **join** $\vee$ always exist.

- Example: The **power set** $\mathscr{P}(S)$ (set of all subsets) of a set S is a lattice with:
  - $a \leq b \ \doteq\ a \subseteq b$
  - $a \vee b \ \doteq\ a \cup b$
  - $a \wedge b \ \doteq\ a \cap b$
  - $\top = S$
  - $\bot = \emptyset$

$$
\begin{tikzcd}
 & & S & & \\
 & & a \cup b \cup c \arrow[u, phantom, sloped, "\subseteq"] & & \\
 & a \cup b \arrow[ur, phantom, sloped, "\subseteq"] & & b \cup c \arrow[ul, phantom, sloped, "\supseteq"] & \\
a \arrow[ur, phantom, sloped, "\subseteq"] & & b \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & & c \arrow[ul, phantom, sloped, "\supseteq"] \\
 & a \cap b \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & & b \cap c \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & \\
 & & a \cap b \cap c \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & & \\
 & & \emptyset \arrow[u, phantom, sloped, "\subseteq"] & &
\end{tikzcd}
$$

In a lattice, meet and join are product and co-product

- If you see a lattice as a category:
  - the **meet** $\wedge$ is the **product** in the category.
  - the **join** $\vee$ is the **co-product** in the category.

$$
\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "f \times g"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"] \\
 & A + B \arrow[d, "f+g"] & \\
 & P &
\end{tikzcd}
$$

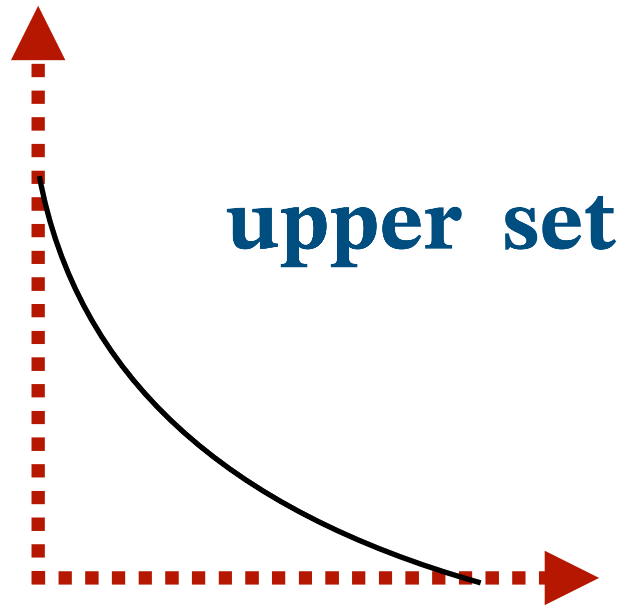
$$
\frac{f : \dfrac{X}{A} \qquad g : \dfrac{X}{B}}{f \times g : \dfrac{X}{A \wedge B}}
\qquad\qquad
\frac{f : \dfrac{A}{P} \qquad g : \dfrac{B}{P}}{f + g : \dfrac{A \vee B}{P}}
$$

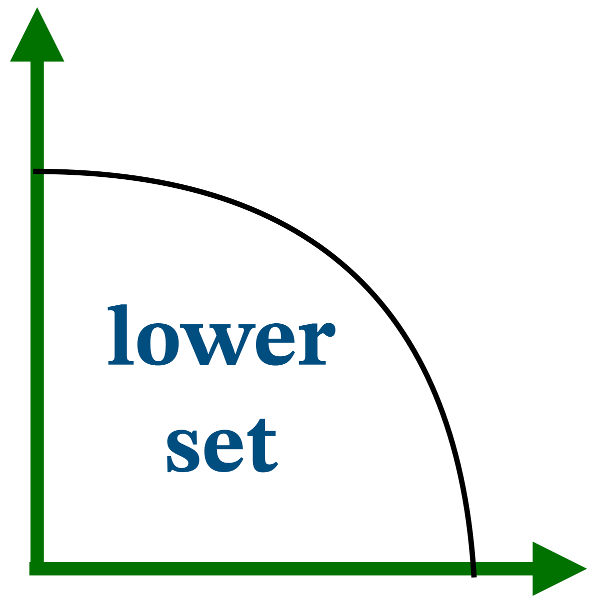
$$
\pi_A : \frac{A \wedge B}{A} \qquad \pi_B : \frac{A \wedge B}{B} \qquad\qquad i_A : \frac{A}{A \vee B} \qquad i_B : \frac{B}{A \vee B}
$$

Upper/lower sets

- An **upper set** is a subset of a poset such that, if x is in the poset, all "higher" elements are.
- A **lower set** is a subset of a poset such that, if x is in the poset, all "lower" elements are.

$$\frac{x \in U \qquad x \leq y}{y \in U} \qquad\qquad \frac{x \in L \qquad y \leq x}{y \in U}$$





- Let $\mathbf{U}P$ and $\mathbf{L}P$ be the sets of all upper/lower sets. These are posets choosing:

$$\langle \mathbf{U}P, \supseteq \rangle \qquad \top = \emptyset \qquad \bot = P$$

$$\langle \mathbf{L}P, \subseteq \rangle \qquad \top = P \qquad \bot = \emptyset$$

Upper / lower closure

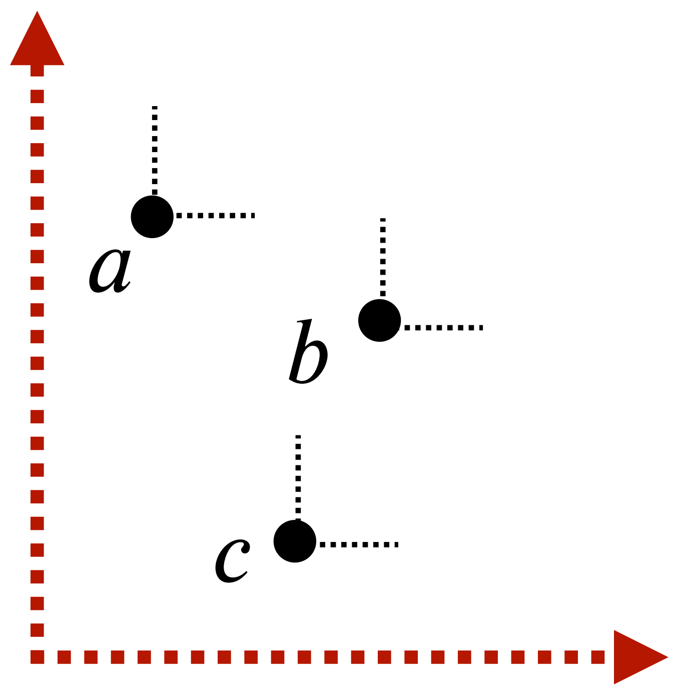
- The **upper closure** of an element is the set of elements that dominate it:

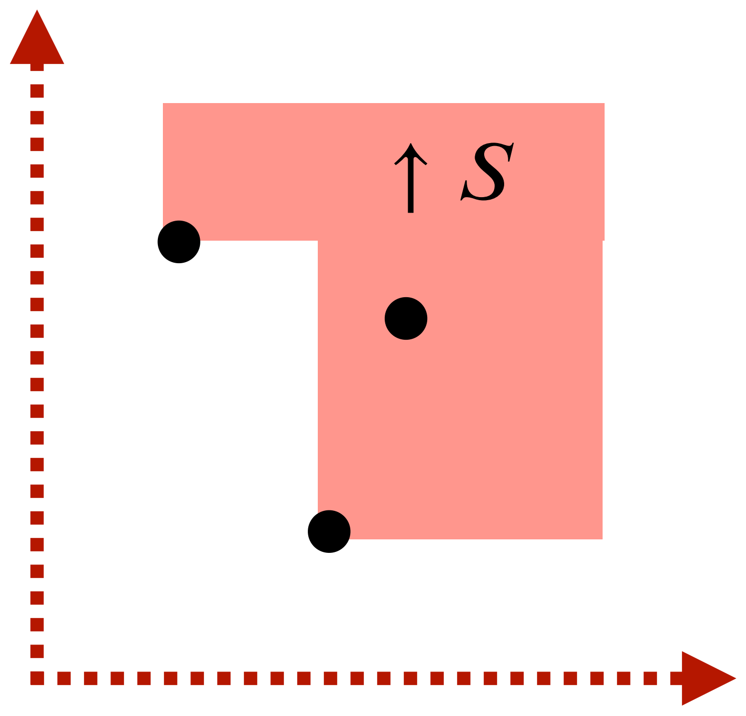
$$\uparrow a \doteq \{x \in P : a \leq x\}$$

- Upper closure of a set:

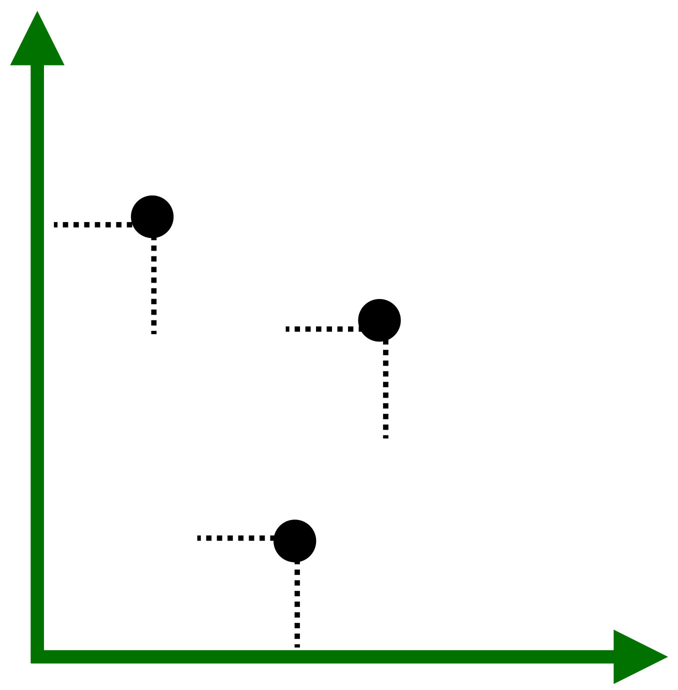
$$\uparrow S \doteq \cup_{a \in S} \uparrow a$$

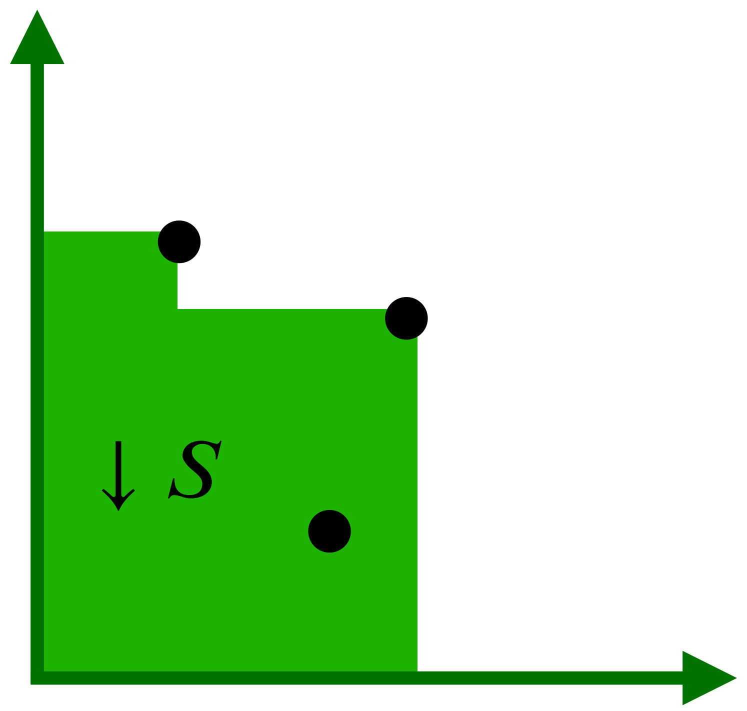
$S = \{a, b, c\}$





- **Lower closure** is defined symmetrically:





Upper / lower closure

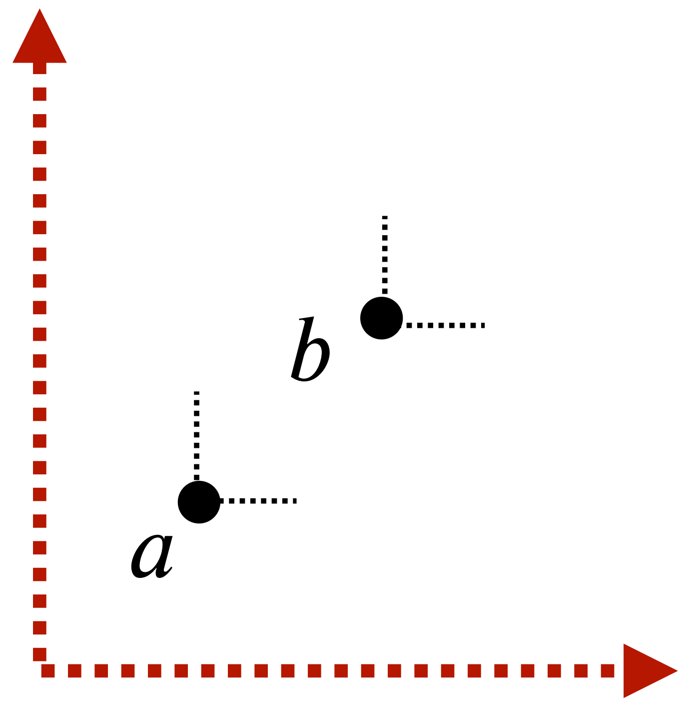
- The **upper closure** of an element is the set of elements that dominate it:

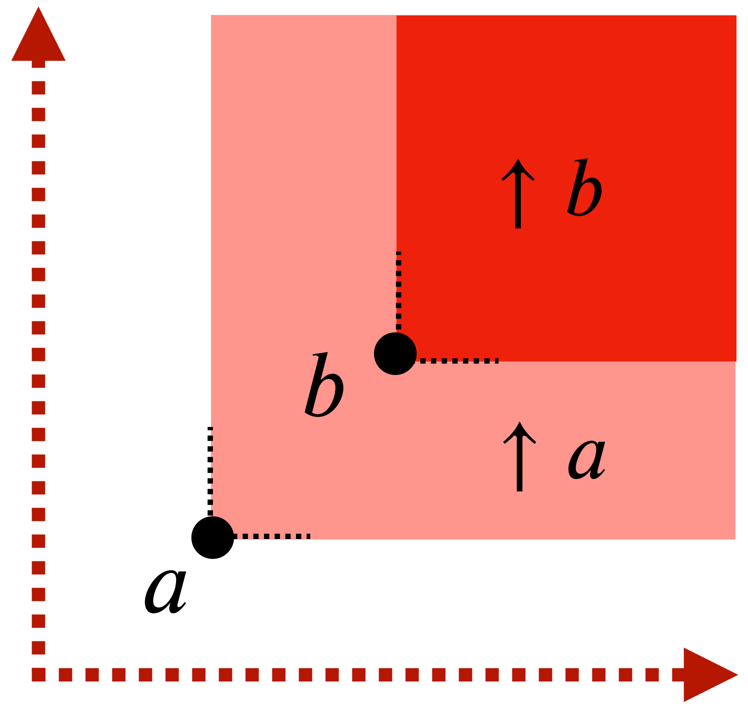
$$\uparrow a \doteq \{x \in P \;:\; a \leq x\}$$

- Upper closure is a **monotone function:**

$$\langle P, \leq_P \rangle \xrightarrow{\;\;\uparrow\;\;} \langle \mathbf{U}P, \supseteq \rangle$$

$$\frac{a \leq_P b}{\uparrow a \supseteq \uparrow b}$$





Functors

- A **functor** $F : C \to D$ between two categories is defined by two maps:
  - a map from objects to objects $f_0 : \mathrm{Ob}_C \to \mathrm{Ob}_C$
  - maps from hom-sets to hom-sets $f_{ab} : \mathrm{Hom}(a; b) \to \mathrm{Hom}(f_0(a); f_0(b))$
- (We overload the notation and write F for both functions.)
- These maps need satisfy two conditions:
  - Identities map to identities: $F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$
  - Composition is respected: $F(g \mathbin{;} h) = F(g) \mathbin{;} F(h)$

C

D

Monotone functions as functors

- A **functor** $F : C \to D$ need satisfy two conditions:
  - Identities map to identities: $F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$
  - Composition is respected: $F(g \mathbin{;} h) = F(g) \mathbin{;} F(h)$

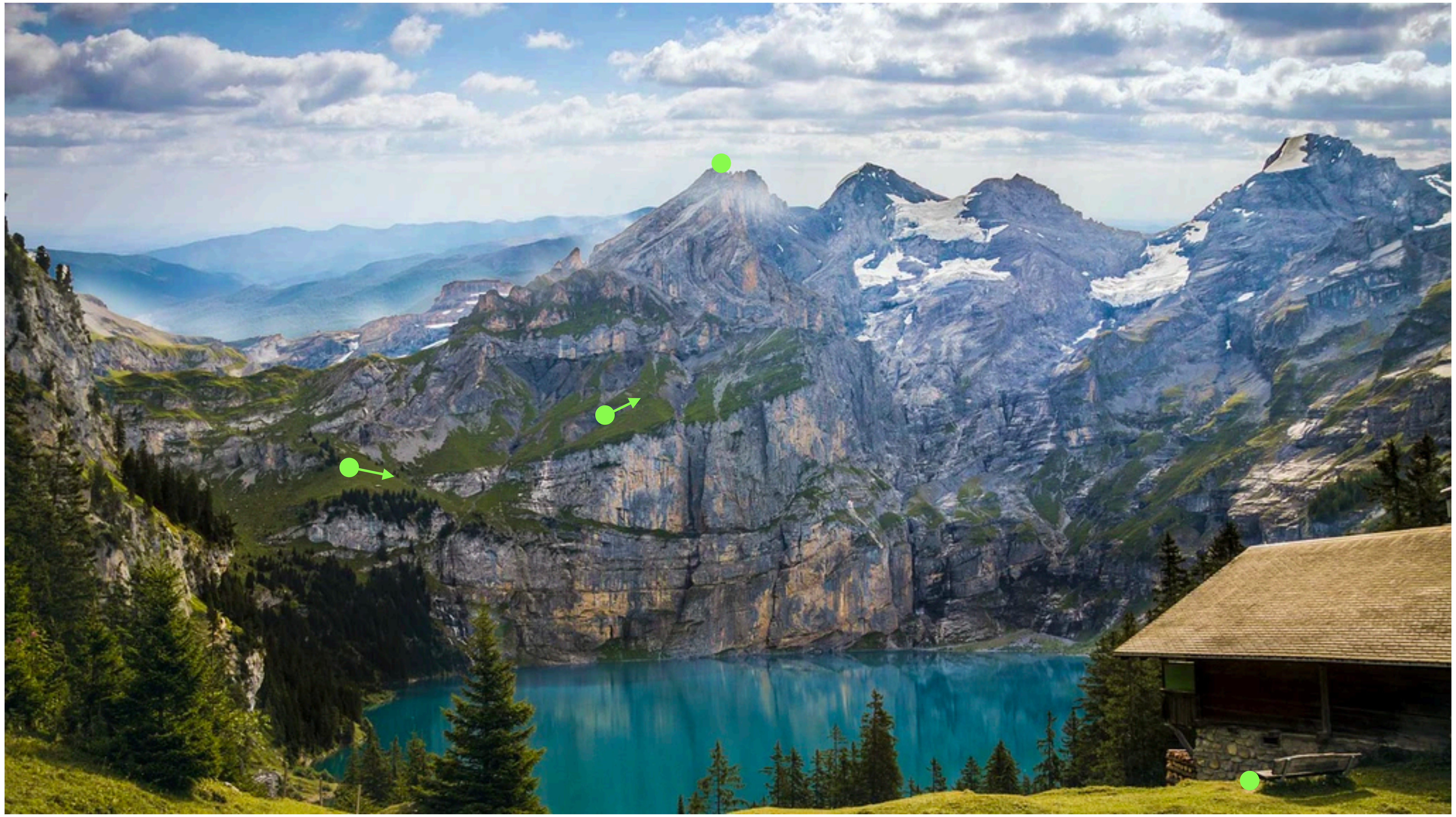
- A **function** $f : \langle P, \leq_P \rangle \to \langle Q, \leq_Q \rangle$ is **monotone** iff

$$\frac{a \leq_P b}{f(a) \leq_Q f(b)}$$

$\langle P, \leq_P \rangle$ $\langle Q, \leq_Q \rangle$

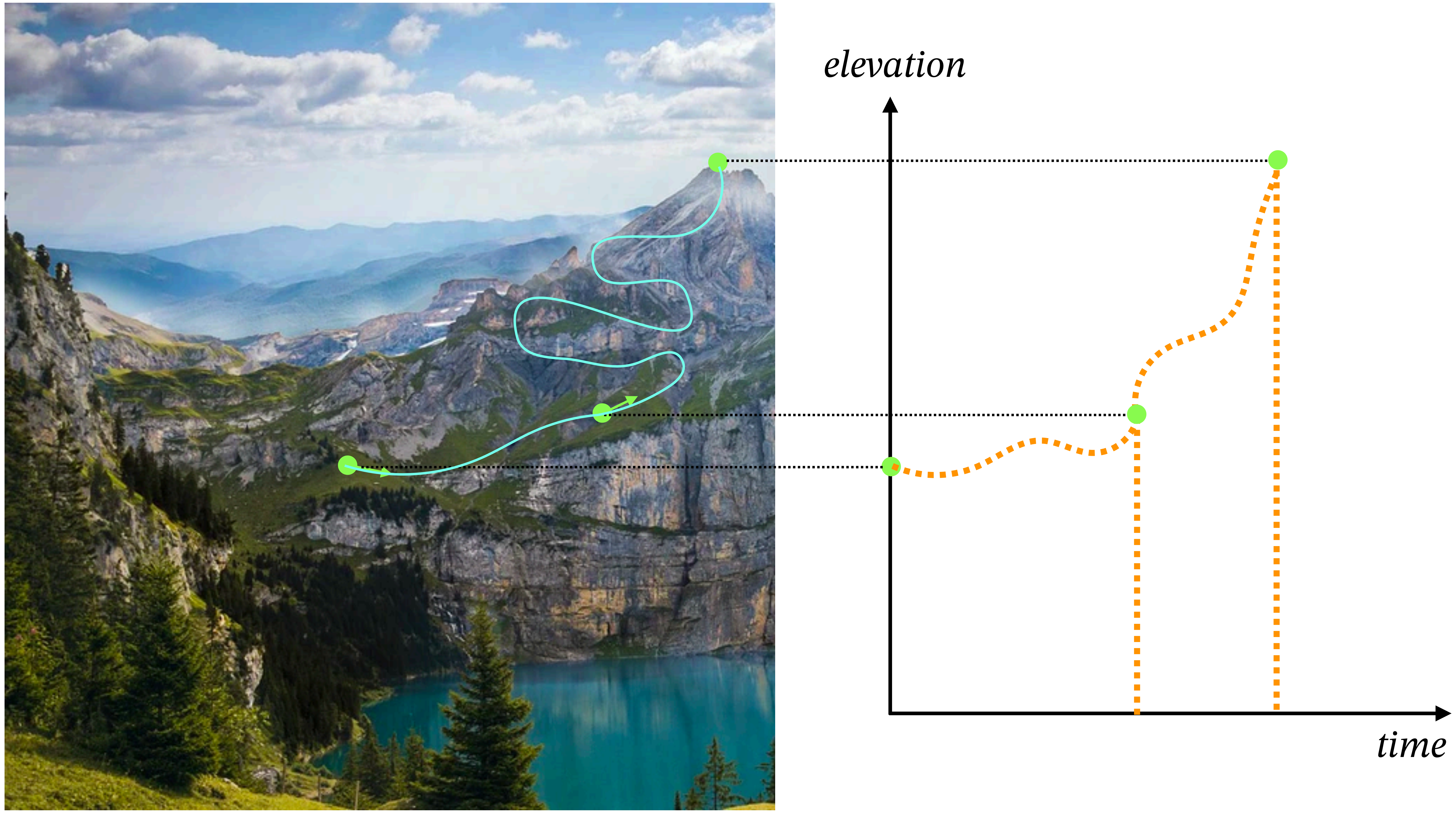
Back to the Swiss Mountains

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.



Elevation as a functor

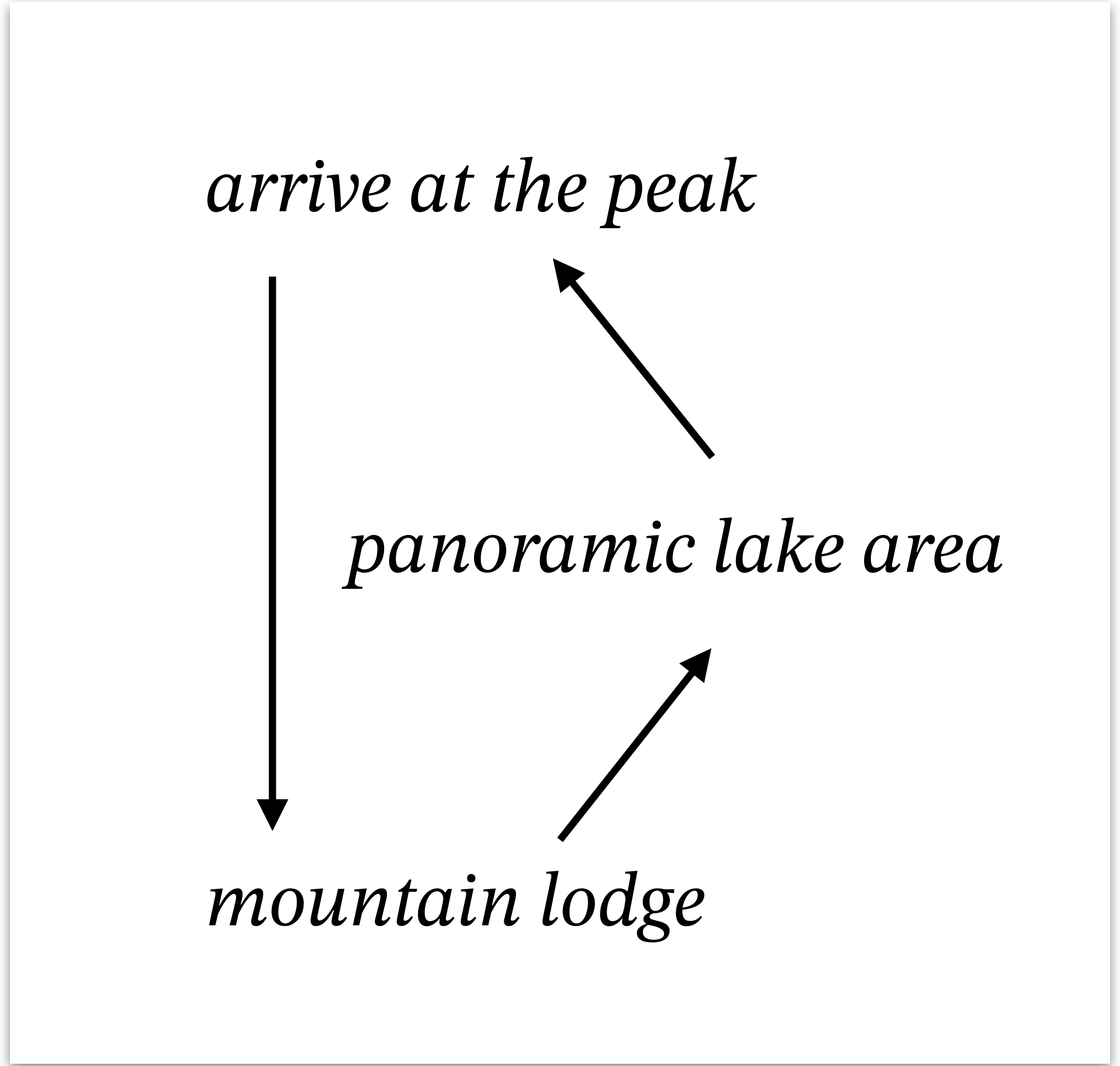
- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.
- There is a functor elevation from Berg to elevation profiles.



Path planning as the search for a functor

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination ≤ 1/2.
- Define a category **Plans** where objects are areas of the mountain and morphisms describe visiting order constraints.

Plans



BergAma

- A **Plan** is a morphism in Plans.
- **Planning** means finding a functor from **Plans** to **BergAma.**

Path planning as the search for a functor

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination ≤ 1/2.
- Define a category **Plans** where objects are areas of the mountains and morphisms describe order constraints.

$$\begin{tikzcd}[column sep=huge]
\mathbf{Plans} \arrow[rr, bend left=50, "\textit{planner 1}"] \arrow[rr, phantom, bend left=25, "\leq"] \arrow[rr, "\textit{planner 2}"] \arrow[rr, phantom, bend right=25, "\leq"] \arrow[rr, bend right=50, "\textit{planner 3}"'] & & \mathbf{BergAma}
\end{tikzcd}$$

- A **Plan** is a morphism in Plans.
- **Planning** means finding a functor from **Plans** to **BergAma.**
- **An optimal planner is one that chooses** the **shortest paths.**

A category of categories

- There exists **a category of (small) categories** called **Cat:**
  - objects are categories
  - morphisms are functors
  - the identities are identity functors

$$\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[r, bend left, "G"] & E \end{tikzcd}$$

- Need to prove that functors compose and that composition is associative:

$$\frac{F : C \to D \qquad G : D \to E}{F \mathbin{;} G : C \to E}$$

$$F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$$

$$F(g \mathbin{;} h) = F(g) \mathbin{;} F(h)$$