Session 3 - Specialization

Applied Compositional Thinking for Engineers

Logistics, announcements

- Tomorrow at **14:00 UTC** we will have a talk from Dr. David Spivak, titled:

  "Applied Category Theory: Towards a science of Interdisciplinarity"

Identity: Attributes, sameness, constraints

Outline of today's lecture:

- Databases, sets, and functions;
- Formal definition of **Set**;
- Formal definition of a subcategory;
- Various examples of subcategories.

Attributes and queries when building a robot

- You are building a robot and choose an **electric motor** from a catalogue:

Motor ID	Company	Size [mm^3]	Weight [g]	Max Power [W]	Cost [USD]
1204	SOYO	20 x 20 x 30	60.0	2.34	19.95
1206	SOYO	28 x 28 x 45	140.0	3.00	19.95
1207	SOYO	35 x 35 x 26	130.0	2.07	12.95
2267	SOYO	42 x 42 x 38	285.0	4.76	16.95
2279	Sanyo Denki	42 x 42 x 31.5	165.0	5.40	164.95
1478	SOYO	56.4 x 56.4 x 76	1,000	8.96	49.95
2299	Sanyo Denki	50 x 50 x 16	150.0	5.90	59.95

- We can use **sets and functions** to think about attributes:

Consider the table with columns $M \times C \times S \times W \times J \times P$. By using

$$M := \{1204, 1206, 1207, 2267, 2279, 1478, 2299\},$$
$$C := \{\text{SOYO}, \text{Sanyo Denki}\},$$

we can define the map $\text{Company} : M \to C$ and e.g. know

$$\text{Company}(1204) = \text{SOYO}.$$

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- We only want motors from company Sanyo Denki:

$$\text{Company}^{-1}(\{\text{Sanyo Denki}\}) = \{2279, 2299\} \subset M$$

- Consider Price : $M \to P$, and just prices between 40 USD and 200 USD:

$$\text{Price}^{-1}(\{49.95, 59.95, 164.95\}) = \{1478, 2299, 2279\} \subset M$$

Operations on tables can be composed

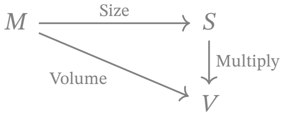
- We want to know the volume of the components:

Define $V = \mathbb{R}_{\geq 0}$ and $S = \mathbb{R}^3_{\geq 0}$. Then:

$$\begin{aligned}\text{Multiply} :\ & S \to V \\ & \langle l, w, h \rangle \mapsto l \cdot w \cdot h.\end{aligned}$$

We know $\text{Size} :\ M \to S$ maps motors to sizes; Hence, $\text{Volume} :\ M \to V$ is given by:

$$\text{Size} \fatsemi \text{Multiply}$$



This is part of a category **Sizes** where:

- $M, S, V \in \text{Ob}_{\mathbf{Sizes}}$;
- Size, Volume, Multiply are morphisms in **Sizes**.

How to include more attributes and functions?

- What if we want to add new morphisms? All the ones of the catalogue?
- This would form a category **Database**. What if we want more?

Definition

The category of sets **Set** is defined by:

1. *Objects*: all sets.
2. *Morphisms*: given sets X and Y, $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the set of all functions from X to Y.
3. *Identity morphism*: given a set X, its identity morphism id_X is the identity function $X \to X,\ \mathrm{id}_X(x) = x$.
4. *Composition operation*: the composition operation is function composition.

- **Exercise:** prove that **Set** is a category.
- What if I want to include some **some sets/functions** and not all?
- In **Database**, I need to make sure that:
  - For any function in **Database**, sources and targets are in **Database**;
  - Compositions of functions in **Database** are in **Database**.

How to consider restricted sets of objects/morphisms?

Definition

A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is a category for which:

1. All the objects in $\mathrm{Ob}_{\mathbf{D}}$ are in $\mathrm{Ob}_{\mathbf{C}}$;
2. For any objects $A, B \in \mathrm{Ob}_{\mathbf{D}}$, $\mathrm{Hom}_{\mathbf{D}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{C}}(A, B)$;
3. If $A \in \mathrm{Ob}_{\mathbf{D}}$, then $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ is in $\mathrm{Hom}_{\mathbf{D}}(A, A)$;
4. If $f : A \to B$ and $g : B \to C$ in $\mathbf{D}$, then the composite $f \mathbin{⨾} g$ in $\mathbf{C}$ is in $\mathbf{D}$ and represents the composite in $\mathbf{D}$.

- **FinSet** is a subcategory of **Set**:
  - **FinSet** is the category of finite sets and all functions between them.
  - While an object $X \in \mathrm{Ob}_{\mathbf{Set}}$ is a set with arbitrary cardinality, $\mathrm{Ob}_{\mathbf{FinSet}}$ only includes sets X' with finitely many elements, i.e., with cardinality $n \in \mathbb{N}$.
  - Objects of **FinSet** are in **Set**, but the converse is not true. Furthermore, given $X, Y \in \mathrm{Ob}_{\mathbf{FinSet}}$, we know that $X, Y \in \mathrm{Ob}_{\mathbf{Set}}$ and $\mathrm{Hom}_{\mathbf{FinSet}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(X, Y)$.

Set is a subcategory of **Rel**

Definition

Let X, Y be sets. A relation $R \subseteq X \times Y$ is a **function** $f : X \to Y$ if:

- $\forall\, x \in X \quad \exists\, y \in Y \;:\; \langle x, y\rangle \in R$.
- $\forall\, \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in R$ holds : $\quad x_1 = x_2 \;\Rightarrow\; y_1 = y_2$.

1. In both **Rel** and **Set** objects are all sets;
2. Given $A, B \in \mathrm{Ob}_{\mathbf{Set}}$, we know that $\mathrm{Hom}_{\mathbf{Set}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{Rel}}(A, B)$;
3. For each $A \in \mathrm{Ob}_{\mathbf{Set}}$, the identity relation $\mathrm{id}_A = \{\langle a, a'\rangle \in A \times A \mid a = a'\}$ is the identity function $\mathrm{id}_A : A \to A$;
4. Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations which are functions. Then their composition $R \,⨾\, S \subseteq X \times Z$ is again a function:
   - Since R is a function, there exists a $y \in Y$ s.t. $\langle x, y\rangle \in R$. Then, since S is a function, there exists a $z \in Z$ s.t. $\langle y, z\rangle \in S$. By definition of composition of relations, $\langle x, z\rangle \in R \,⨾\, S$.
   - Let $\langle x_1, z_1\rangle, \langle x_2, z_2\rangle \in R \,⨾\, S$. Suppose $x_1 = x_2$. There exists $y_1, y_2 \in Y$ s.t. $\langle x_i, y_i\rangle \in R$ and $\langle y_i, z_i\rangle \in S$, $i \in \{1, 2\}$. Since R is a function, $x_1 = x_2$ implies $y_1 = y_2$, which implies $z_1 = z_2$ (because S is a function).

InjSet is a subcategory of **Set**

Definition

Let X, Y be sets. A function $f : X \to Y$ is *injective* if, $\forall x, x' \in X$: $f(x) = f(x') \implies x = x'$.

The category **InjSet** has:

- Objects: all sets;
- Morphisms: injective functions;
- Identity and composition: as in **Set**.

1. **InjSet** has the same objects as **Set**;
2. For any $A, B \in \mathrm{Ob}_{\mathbf{InjSet}}$, $\mathrm{Hom}_{\mathbf{InjSet}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{Set}}(A, B)$;
3. Given $A \in \mathrm{Ob}_{\mathbf{InjSet}}$ the identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{Set}}(A, A)$ is the identity in $\mathrm{Hom}_{\mathbf{InjSet}}(A, A)$ (the identity function is injective);
4. Given morphisms $f : A \to B$ and $g : B \to C$ in **InjSet**, their composition in **Set** is in **InjSet**, i.e. the composition of injective functions is injective:

$$
\begin{aligned}
(f \fatsemi g)(a) = (f \fatsemi g)(a') &\implies f(a) = f(a') \\
&\implies a = a',
\end{aligned}
$$

The category of swiss mountains **Berg**

Let **Berg** be the category defined as follows:

- Objects are tuples $\langle p, v \rangle$, where
  - $p \in L$ (locations),
  - $v \in \mathbb{R}^3$ (tangent vector to L at p, velocities).
- A morphism $\langle p_1, v_1 \rangle \to \langle p_2, v_2 \rangle$ is $\langle \gamma, T \rangle$, where
  - $T \in \mathbb{R}_{\geq 0}$ (time),
  - $\gamma : [0, T] \to L$ is a C^1 function with $\dot{\gamma}(0) = v_1$ and $\dot{\gamma}(T) = v_2$ (we take one-sided derivatives at the boundaries).
- For any object $\langle p, v \rangle$, we define its identity morphism $1_{\langle p, v \rangle} = \langle \gamma, 0 \rangle$ formally: its path γ is defined on the closed interval $[0, 0]$, i.e. $T = 0$ and $\gamma(0) = p$. We declare this path to be C^1 by convention, and declare its derivative at 0 to be v.
- Concatenation: given morphisms $\langle \gamma_1, T_1 \rangle : \langle p_1, v_1 \rangle \to \langle p_2, v_2 \rangle$ and $\langle \gamma_2, T_2 \rangle : \langle p_2, v_2 \rangle \to \langle p_3, v_3 \rangle$, their composition is $\langle \gamma, T \rangle$ with $T = T_1 + T_2$ and

$$\gamma(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_1 \\ \gamma_2(t - T_1) & T_1 \leq t \leq T_1 + T_2. \end{cases}$$

The category of swiss mountains **Berg**

- **Exercise**: **Berg** is a category:

The category of swiss mountains **BergAma**?

- For each path, we can compute the steepness;
- As amateur hikers, we just want to consider paths with a certain maximum inclination, in $(-1, 1)$;
- By taking the absolute value, we obtain a function:

$$\text{MaxSteepness} : \text{Hom}_{\mathbf{Berg}}(\langle p_1, v_1 \rangle, \langle p_2, v_2 \rangle) \longrightarrow [0, 1).$$

- In **BergAma**, we just consider paths which have a maximal steepness $< 1/2$.
- Is **BergAma** a subcategory of **Berg**?

1. $\text{Ob}_{\mathbf{Berg}} \supset \text{Ob}_{\mathbf{BegAma}}$ (steep objects out!);
2. For any $A, B \in \text{Ob}_{\mathbf{BergAma}}$, we know $\text{Hom}_{\mathbf{BergAma}} \subseteq \text{Hom}_{\mathbf{Berg}}$;
3. Given the restriction on objects, the identity morphisms in **Berg** do not violate the steepness constraint, and they are identity morphisms in **BergAma**;
4. Given two morphisms f, g which can be composed in **BergAma**, the maximum steepness of their composition $f \fatsemi g$ is:

$$\text{MaxSteepness}(f \fatsemi g) = \max \{\text{MaxSteepness}(f), \text{MaxSteepness}(g)\} < 1/2.$$

The category of swiss mountains **BergLazy**?

- For each path, we can compute the length in meters;
- As amateur hikers, we just want to consider paths which are shorter than 1 km;
- The composition of two morphisms in **BergLazy** of length 0.6 km each result in a 1.2 km long path, hence not in **BergLazy**.

The category **Draw** of drawings

Definition

There exists a category **Draw** in which:

1. An object in $\alpha \in \mathrm{Ob}_{\mathbf{Draw}}$ is a black-and-white drawing, that is a function $\alpha : \mathbb{R}^2 \to \texttt{Bool}$.
2. A morphism in $\mathrm{Hom}_{\mathbf{Draw}}(\alpha, \beta)$ between two drawings α and β is an invertible map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\alpha(x) = \beta(f(x))$.
3. The identity function at any object α is the identity map on $\mathbb{R}^2$.
4. Composition is given by function composition.

- **Exercise:** Check whether just considering
  - affine invertible transformations,
  - rototranslations,
  - scalings,
  - translations,
  - rotations,

form a subcategory of **Draw**.