Session 4 - Combining

Applied Compositional Thinking for Engineers

Plan

- Isomorphisms and sameness
- Products
- Examples for the "universal property"

Isomorphisms and sameness

We want to be able to say when a morphism is "**invertible**". Such morphisms will be called **isomorphisms**.

Let's look at the situation with functions. An invertible function is called "bijective". Two different formulations of the the definition:

Version 1: "$f: X \to Y$ is bijective if, for every $y \in Y$ there exists precisely one $x \in X$ such that $f(x) = y$;

Version 2: "$f: X \to Y$ is bijective if there exists a function $g: Y \to X$ such that $f \mathbin{⨟} g = id_X$ and $g \mathbin{⨟} f = id_Y$".

Definition: Let $\mathbf{C}$ be a category, let X and Y be objects.
A morphism $f\colon X \to Y$ is an **isomorphism** if there exists a morphism $g\colon Y \to X$ such that $f \fatsemi g = \mathrm{id}_X$ and $g \fatsemi f = \mathrm{id}_Y$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Remark: The morphism g is called the **inverse** of f; if such g exists it is uniquely determined.

Definition: Objects X and Y of $\mathbf{C}$ are **isomorphic** if there exists an isomorphism $X \to Y$ or $Y \to X$.

Example: These sets are all isomorphic:

$$\{0,1\},\ \{\text{true},\ \text{false}\},\ \{\perp,\top\},\ \{\text{left},\ \text{right}\},\ \{-,+\},\ \{*,\dagger\}$$

$$\begin{tikzcd} \{0,1\} \arrow[r, bend left, "f"] & \{\text{true},\ \text{false}\} \arrow[l, bend left, "g"] \end{tikzcd}$$

These sets are "**interchangeable**". Often we want to **keep track** of the isomorphism we use to interchange them !

Example: Currency exchangers

$$E_{a,b} : \begin{cases} \mathbb{R} \times \{\mathrm{USD}\} & \longrightarrow \mathbb{R} \times \{\mathrm{EUR}\} \\ \langle x, \mathrm{USD} \rangle & \longmapsto \langle ax - b, \mathrm{EUR} \rangle \end{cases}$$

$$\begin{tikzcd} \mathrm{USD} \arrow[rr, bend left, "E_{a,b}"] & & \mathrm{EUR} \arrow[ll, bend left, "E'_{a',b'}"] \end{tikzcd}$$

Products

The notion of *product* in category theory generalizes the notion of *cartesian product* of sets.

Recall: For sets X and Y:

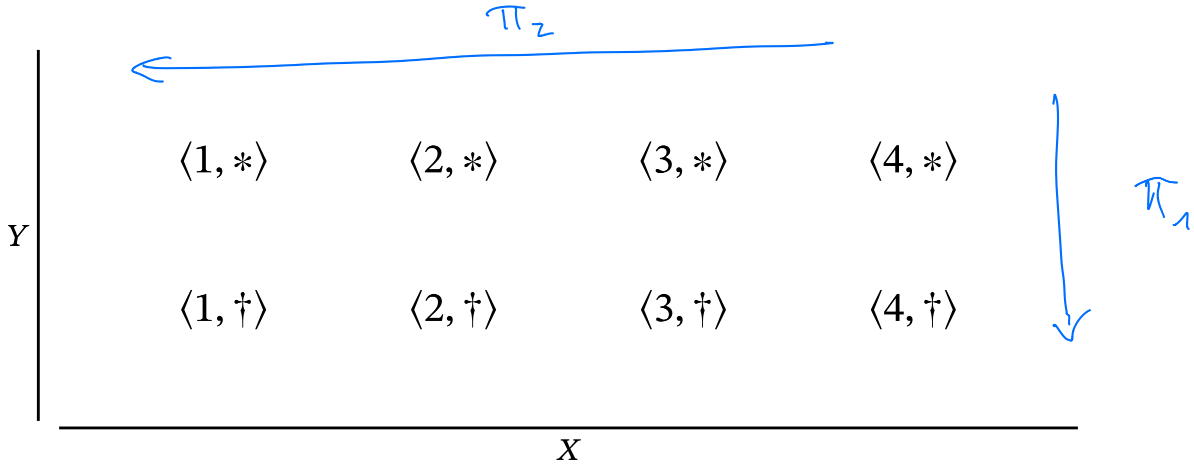
$$X \times Y = \{\langle x, y\rangle \mid x \in X,\ y \in Y\}$$

Example:

$$X = \{1, 2, 3, 4\}, \qquad Y = \{\dagger, *\}$$

$$X \times Y = \{\langle 1, \dagger\rangle, \langle 2, \dagger\rangle, \langle 3, \dagger\rangle, \langle 4, \dagger\rangle, \langle 1, *\rangle, \langle 2, *\rangle, \langle 3, *\rangle, \langle 4, *\rangle\}$$

The cartesian product comes with "projection maps" included in the package:



Projection maps:

$$X \xleftarrow{\ \pi_1\ } X \times Y \xrightarrow{\ \pi_2\ } Y$$

$$x \longmapsfrom \langle x, y \rangle \longmapsto y$$

Direct sum of vector spaces: Let V and W be vector spaces. Their direct sum is $V \oplus W = \{\langle v, w\rangle \mid v \in V,\ w \in W\}$.

$$\mathbb{R}^3 \longleftarrow \mathbb{R}^3 \oplus \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

"Min" in an ordered set: Consider $(\mathbb{R}, \leq)$; draw an arrow $x_1 \to x_2$ if $x_1 \leq x_2$.

$$2.5 \longleftarrow \min\{2.5, 3.3\} \longrightarrow 3.3$$

Greatest common divisor: Let $m, n \in \mathbb{N}$. Draw an arrow to indicate "divides". E.g. since $6 \mid 12$ we would draw $6 \longrightarrow 12$.

$$12 \longleftarrow gcd\{12, 18\} \longrightarrow 18$$

Intersection of subsets: Let S be a set, and $X, Y \subseteq S$ subsets. Draw an arrow to indicate subset inclusion.

E.g. $S = \{a, b, c, d\}, X = \{a, b, c\}, Y = \{b, c, d\}$.

$$X \longleftarrow X \cap Y \longrightarrow Y$$

What's the general definition !!??

Let's look at an "engineering example" to get some intuition.

Suppose you are at an engineering conference in Switzerland, and there will be a hike as a group outing...

The organizers have prepared **snacks** to go. Each participant can choose a **food** from

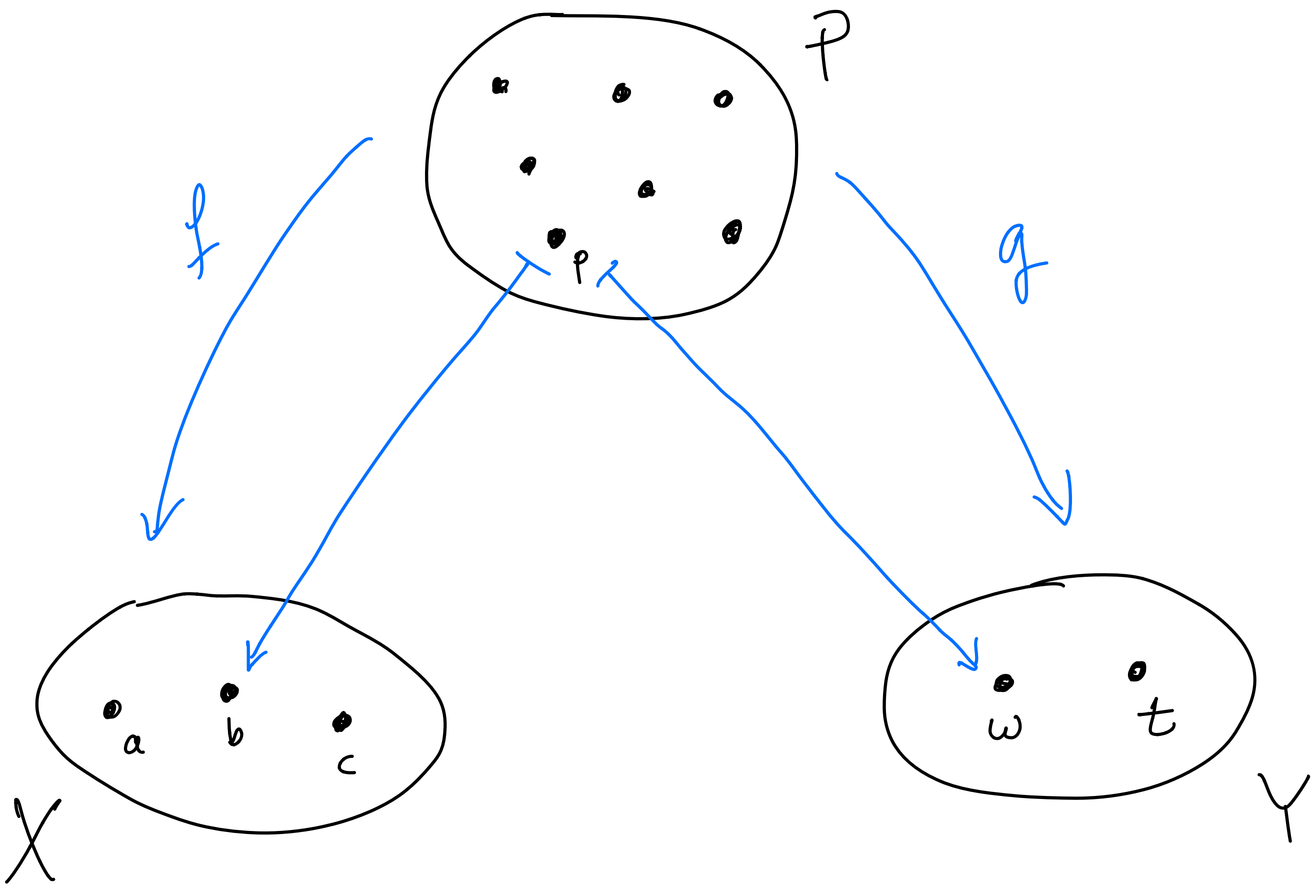
$$X = \{a, b, c\} \qquad \text{(“apple”, “banana”, “carrot”)}$$

and a **drink** from

$$Y = \{w, t\} \qquad \text{(“water”, “tea”)}$$

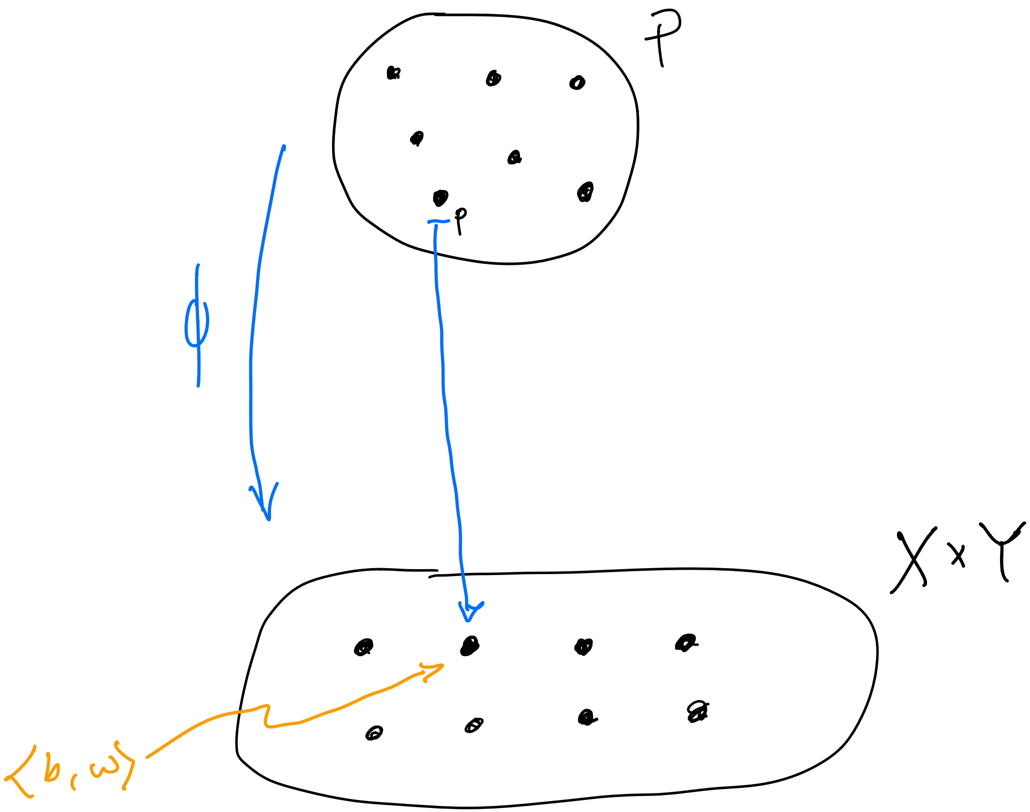
Let P denote the set of participants.

The distribution/choosing of snacks could be organized like this: each participant chooses a food, and chooses a drink.

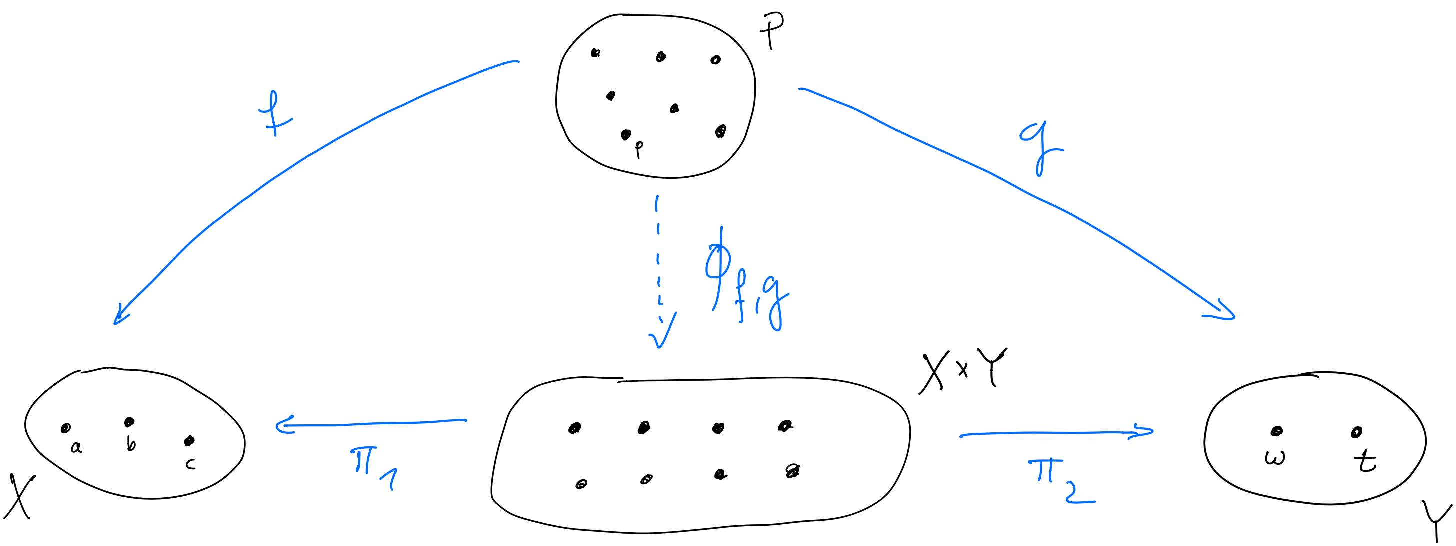


Or, snacks could be prepackaged.
All possible **combinations** of food and drink choices: $X \times Y$.

Now a participant just makes one choice about which lunch package:



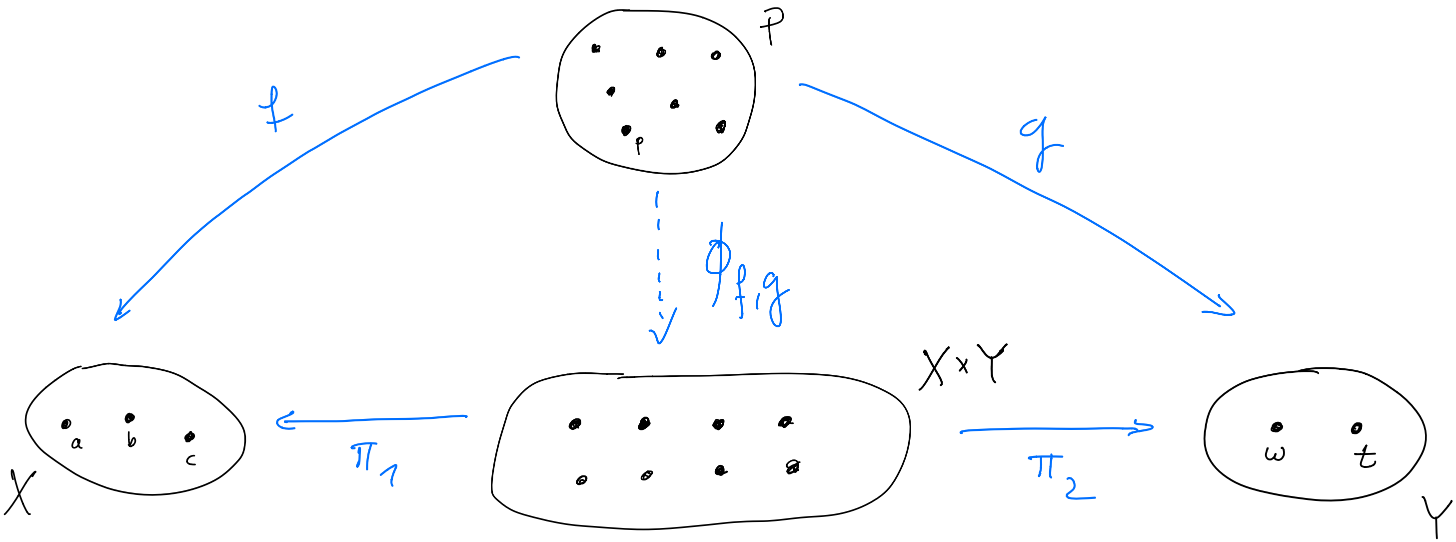
In which sense are the two approaches essentially the same?



Given f and g, we can build $\phi_{f,g}$:

$$\phi_{f,g} : P \longrightarrow X \times Y, \quad p \longmapsto \langle f(p), g(p) \rangle.$$

In which sense are the two approaches essentially the same?



Given $\phi_{f,g}$, we can recover f and g:

$$f = \phi_{f,g} \fatsemi \pi_1 \qquad \text{and} \qquad g = \phi_{f,g} \fatsemi \pi_2.$$

The diagram is **commutative**!

This state of affairs actually characterizes what a product is…

🤔

We will see: a product is defined *“up to isomorphism”*….

🧐

Definition: Let **C** be a category, and let X and Y be objects. A **product** of X and Y consists of:

- Data:
  1. An object Z (this is "the" product)
  2. Morphisms $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$

- Rule: ("universal property of the product")
  $\forall\, P \in \mathrm{Ob}_{\mathbf{C}},\ \forall\, f : P \to X,\ \forall\, g : P \to Y,\ \exists!\, \phi_{f,g} : P \to Z$, s.t.

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d, "\phi_{f,g}"] \arrow[dr, "g"] & \\ X & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commutes.

Remarks:

- Products do not always exist! (E.g. number fields.)

- Strictly speaking, a product consists of an object and the two projection morphisms, but...

- There may be different constituent data that satisfy the definition for "product of X and Y", e.g.

$$X \xleftarrow[\pi_1]{} Z \xrightarrow[\pi_2]{} Y \quad \text{and} \quad X \xleftarrow[\tilde{\pi}_1]{} \tilde{Z} \xrightarrow[\tilde{\pi}_2]{} Y$$

  One can prove: in such a case, $Z \simeq \tilde{Z}$.
  (And in a way compatible with all the projection morphisms).

  Hence we speak of "the" product of X and Y, and write "$X \times Y$".

Slogan: the product of X and Y is the

"most efficient way" to have both X ***and*** Y.

Examples for the universal property

Direct sum of vector spaces: Let V and W be vector spaces. Their direct sum is $V \oplus W = \{\langle v, w\rangle \mid v \in V,\ w \in W\}$.

$$\begin{tikzcd}
& \mathbb{R}^7 \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
\mathbb{R}^3 & \mathbb{R}^3 \oplus \mathbb{R}^2 \arrow[l] \arrow[r] & \mathbb{R}^2
\end{tikzcd}$$

"Min" in an ordered set: Consider $(\mathbb{R}, \leq)$; draw an arrow $x_1 \to x_2$ if $x_1 \leq x_2$.

$$\begin{tikzcd}
& 1.2 \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
2.5 & \min\{2.5, 3.3\} \arrow[l] \arrow[r] & 3.3
\end{tikzcd}$$

Greatest common divisor: Let $m, n \in \mathbb{N}$. Draw an arrow to indicate "divides". E.g. since $6 \mid 12$ we draw $6 \to 12$.

$$\begin{tikzcd}
& 3 \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
12 & gcd\{12, 18\} = 6 \arrow[l] \arrow[r] & 18
\end{tikzcd}$$

Intersection of subsets: Let S be a set, and $X, Y \subseteq S$ subsets. Draw an arrow to indicate subset inclusion.

$S = \{a, b, c, d\}$, $X = \{a, b, c\}$, $Y = \{b, c, d\}$, $X \cap Y = \{b, c\}$.
Consider also $W = \{c\}$.

$$
\begin{tikzcd}
 & W \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
X & X \cap Y \arrow[l] \arrow[r] & Y
\end{tikzcd}
$$

Example: Two different representations of “the same” product.

Suppose we are a manufacturer and we want to label our products with
- production date (8-digit code)
- model number (4-digit code)

Products

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Dates

Model numbers

Instead of two separate labels, we make one:

$$\underbrace{20210115}_{\text{date}}\underbrace{5900}_{\text{model number}}$$

Call this the "product code".

Set Z = set of all product codes.

$$\begin{tikzcd}
 & P \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
X & Z \arrow[l, "\pi_1", "\text{first 8}"'] \arrow[r, "\pi_2"', "\text{last 4}"] & Y
\end{tikzcd}$$

The set Z, together with the maps

$$X \xleftarrow[\text{first 8}]{\pi_1} Z \xrightarrow[\text{last 4}]{\pi_2} Y$$

will satisfy the definition of "product of X and Y"

even though Z is not precisely the cartesian product $X \times Y$.

Elements of Z are 12 digit codes, while elements of $X \times Y$ are *pairs* $\langle x, y \rangle$ where x is a 8-digit code and y is a 4-digit code.

But: Z and $X \times Y$ are both "the" product of X and Y, so they are isomorphic. In fact, isomorphic in a unique way such that this diagram commutes:

$$
\begin{tikzcd}
 & Z \arrow[dl, "\text{first } 8"'] \arrow[dr, "\text{last } 4"] \arrow[d, dashed, "\phi_{\text{first } 8, \text{last } 4}"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}
$$

$$
\phi_{\text{first } 8, \text{last } 4} : \begin{cases} Z & \longrightarrow & X \times Y \\ 202101155900 & \longmapsto & \langle 20210115, 5900 \rangle \end{cases}
$$