Session 5 - Choosing: Coproducts

Applied Compositional Thinking for Engineers

Logistics, announcements

Week plan:

- **Monday, January 18th at 18:00 UTC**: Session 5 (Combining);
- **Wednesday, January 20th at 14:00 UTC**: Session 6 (Trade-offs);
- **Thursday, January 21st at 18:00 UTC**: Guest Lecture 2 (Dr. Brendan Fong);
- **Friday, January 22nd at 14:00 UTC**: Session 7 (Life is hard);
- **Saturday, January 23rd**: Office/social hours:
  - **09:00-10:00 UTC**;
  - **18:00-19:00 UTC**;
- **Sunday, January 24th**: Office hour:
  - **14:00-15:00 UTC**.

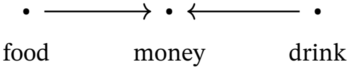
Choosing: coproducts

Outline of today's lecture:

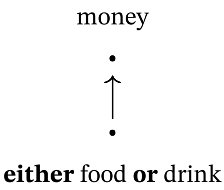
- Intuition behind coproducts;
- Formal definition of coproduct;
- Various examples of coproducts.

Intuition behind coproducts

- Last week you saw **products**;
- Today, we see **coproducts**;
- Consider a vending machine:



- We want to have a notion of:

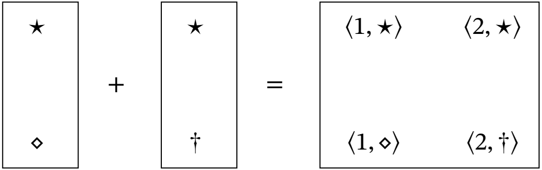


Intuition behind coproducts

- The notion of *coproduct* in category theory generalizes the notion of *disjoint union* of sets.
- Given two sets A, B, their disjoint union is:

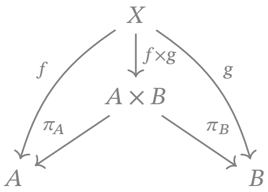
$$A + B = \{\langle 1, a\rangle \mid a \in A\} \cup \{\langle 2, b\rangle \mid b \in B\}.$$

- Consider $\{\star, \diamond\}$ and $\{\star, \dagger\}$:



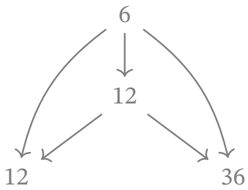
Refresher: Products

- Recall: We have seen the **product** $A \times B$ between objects A and B:

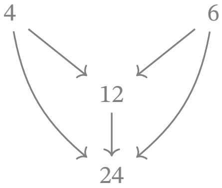


Example: greatest common divisor

- Let $A, B \in \mathbb{N}$. Draw an arrow between A and B if A divides B, e.g. $6 \to 12$.
- Product (greatest common divisor):

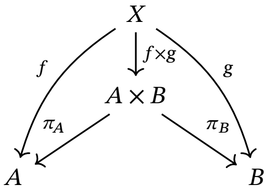


- Coproduct (least common multiple):

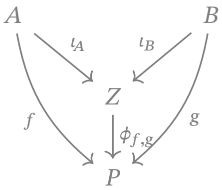


Coproduct, intuitively

- In the coproduct, we reverse the arrows;
- Projections become injections/inclusions;
- Product:



- Coproduct:



Formal definition of coproduct

Definition

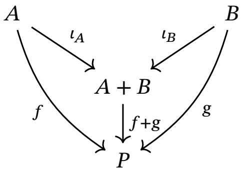
Let $\mathbf{C}$ be a category and let $A, B \in \mathrm{Ob}_{\mathbf{C}}$. The *coproduct* of A and B is:

- an object $A + B \in \mathrm{Ob}_{\mathbf{C}}$, together with
- two *inclusion morphisms* $\iota_A : A \to A + B$ and $\iota_B : B \to A + B$,

such that, given any $P \in \mathrm{Ob}_{\mathbf{C}}$ and morphisms $f : A \to P, g : B \to P$, there exists a *unique* morphism $(f + g) : A + B \to P$ such that

$$f = \iota_A \fatsemi (f + g) \quad \text{and} \quad g = \iota_B \fatsemi (f + g).$$

Diagrammatically:



Example of batteries

- Let's consider two battery producers A and B, each producing specific technologies;

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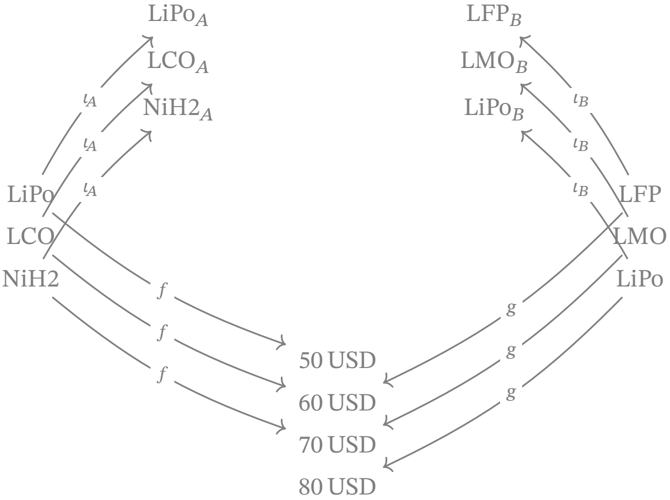


Example of batteries

Example of batteries

- The universal property of coproducts says that there is a **unique** function

$$f + g\colon A + B \to P$$

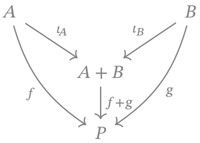
s.t.

$$\iota_A \fatsemi (f+g) = f \text{ and } \iota_B \fatsemi (f+g) = g$$

- Any $x \in A + B$ is either “from A or from B:

$$\text{either } \exists a \in A : x = \iota_A(a) \text{ or } \exists b \in B : x = \iota_B(b).$$

- By the diagram



we must have

$$(f+g)(x) = \begin{cases} f(x) & \text{if } x = \iota_A(a), \quad a \in A, \\ g(x) & \text{if } x = \iota_B(b), \quad b \in B. \end{cases}$$

Coproducts in **Set** and **FinSet**

- Coproducts here are generalization of **disjoint unions**;
- Given $A, B \in \mathrm{Ob}_{\mathbf{Set}}$, $A + B$ is the disjoint union;
- Injections ι_A, ι_B are **inclusion** maps:

$$\iota_A : A \to A + B$$
$$\iota_B : B \to A + B$$

- $f + g$ is given by:

$$(f+g)(x) = \begin{cases} f(x) & \text{if } x = \iota_A(a), \quad a \in A, \\ g(x) & \text{if } x = \iota_B(b), \quad b \in B. \end{cases}$$

Why not the union in **Set**?

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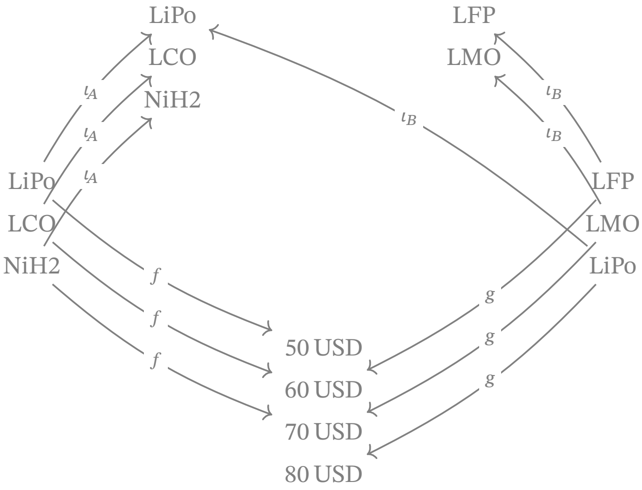


- In general, if $A \cap B \neq \varnothing$, $f + g \colon A \cup B \to P$ cannot exist!
- All elements $x \in A \cap B$ would be sent to $f(x)$ and $g(x)$ for commutativity.

Product and coproduct in **Rel**

- Given $X, Y \in \mathrm{Ob}_{\mathbf{Rel}}$, their coproduct is the **disjoint union** $X + Y$;
- This is equipped with injections:

$$\iota_X : X \to X + Y$$
$$\iota_Y : Y \to X + Y$$

- These induce relations:

$$R_{\iota_X} \subseteq X \times (X + Y)$$
$$R_{\iota_Y} \subseteq Y \times (X + Y).$$

- Note that in **Rel** products and coproducts are closely related: both involve disjoint union of sets.

Join in an ordered set

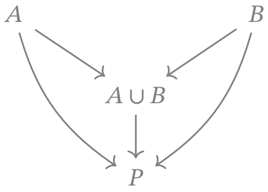
Join in an ordered set

- Consider $\langle \mathbb{R}, \leq \rangle$ and draw $x_1 \to x_2$ if $x_1 \leq x_2$.
- The coproduct of x_1 and x_2 is an element z such that:
  - $x_1 \leq z$;
  - $x_2 \leq z$;
  - For all $x \in \mathbb{R}$ with $x_1 \leq x$ and $x_2 \leq x$, we have $z \leq x$.
- Coproduct is a "least element above both x_1 and x_2" (also called *join*).
- Coproduct of $x_1, x_2 \in \mathbb{R}$ is $\max\{x_1, x_2\}$.

Union of subsets

Union of subsets

Let S be a set, and $X, Y \subseteq S$ subsets. We draw an arrow to indicate subset inclusion.



Direct sum of vector spaces is the product in **Vect**

- There is a category **Vect**, where:
  - Objects: vector spaces;
  - Morphisms: linear maps;
  - Identity morphisms: identity maps;
  - Composition: composition of linear maps;
- Recall: given V and W vector spaces, their **direct sum** is

$$V \oplus W = \{\langle v, w \rangle \mid v \in V, w \in W\}.$$

- Given $\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle \in V \oplus W$, we have:

$$\langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle := \langle v_1 + v_2, w_1 + w_2 \rangle$$

- We showed this is the **product**, but it is also the **coproduct**!

Direct sum of vector spaces is the coproduct in **Vect**

Direct sum of vector spaces is the coproduct in **Vect**

- Injections are:

$$\begin{aligned} \iota_V \colon V &\to V \oplus W & \iota_W \colon W &\to V \oplus W \\ v &\mapsto \langle v, 0_W \rangle & w &\mapsto \langle 0_V, w \rangle \end{aligned}$$

- Let's take any linear maps $S \colon V \to U$, $T \colon W \to U$;
- We need a **unique** $h \colon V \oplus W \to U$ s.t. $S = \iota_V \fatsemi h$ and $T = \iota_W \fatsemi g$;
- For $\langle v, w \rangle \in V \oplus W$, we can write:

$$\begin{aligned} h(\langle v, w \rangle) &= h(\langle v, 0_W \rangle + \langle 0_V, w \rangle) \\ &= h(\iota_V(v) + \iota_W(w)) \\ &= h(\iota_V(v)) + h(\iota_W(v)) \\ &= (\iota_V \fatsemi h)(v) + (\iota_W \fatsemi h)(w) \\ &\overset{!}{=} Sv + Tw \end{aligned}$$

- Hence:

$$\begin{aligned} h \colon V \oplus W &\to U \\ \langle v, w \rangle &\mapsto Sv + Tw \end{aligned}$$

Category of graphs

- One can define the category of graphs **Grph**;
- Objects are graphs $G = \langle V, A, s, t \rangle$, where
  - V is a set of **vertices**;
  - A is a set of **arrows**;
  - $s : A \to V$ is a **source** function;
  - $t : A \to V$ is a **target** function.

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- Morphisms are **graph homomorphisms**;
- Given graphs $G = \langle V, A, s, t \rangle$, $G' = \langle V', A', s', t' \rangle$, a **graph homomorphism** $f : G \to G'$ is given by $f_0 : V \to V'$ and $f_1 : A \to A'$, such that:

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ V & \xrightarrow{f_0} & V' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f_1} & A' \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\ V & \xrightarrow{f_0} & V' \end{array}$$

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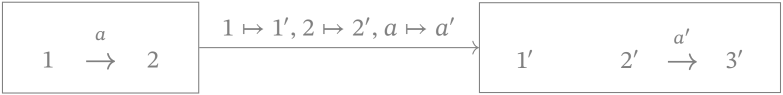
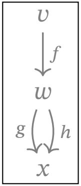
- "Arrows are bound to their vertices"

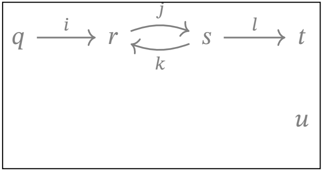


Coproduct in **Grph**

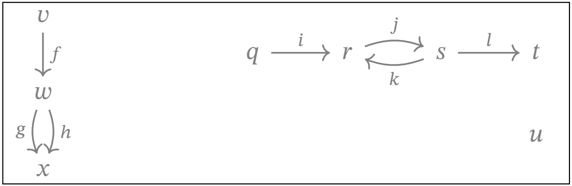
Coproduct in **Grph**

- Consider two graphs:





- Their co-product is:



Coproduct in **Grph**

- Given $G = \langle V, A, s, t \rangle$ and $G' = \langle V', A', s', t' \rangle$, their coproduct is a graph

$$G + G' = \langle V + V', A + A', s + s', t + t' \rangle;$$

- An arrow connects v_1 to v_2 if
  - $v_1, v_2 \in V$ or $v_1, v_2 \in V'$, and
  - an arrow exists in G or G';
- Given $s : A \to V$ and $s' : A' \to V'$, we have (similar for $t + t'$):

$$\begin{aligned} s + s' : A + A' &\to V + V' \\ x &\mapsto \begin{cases} s(x) & \text{if } x \in A \\ s'(x) & \text{if } x \in A'. \end{cases} \end{aligned}$$