Applied Category Theory:
Towards a science of interdisciplinarity

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Outline

A road to true interdisciplinarity

- Scientific disciplines are conceptual analogies of the world.
    - Science: a schematic, conceptual account of phenomena.
    - Engineering is using these accounts to channel world events.
    - But how do different disciplines and accounts cohere?
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  - Interdisciplinarity consists of effective analogy-making.
  - To go further, we need to formalize the analogies themselves.
- Better yet: we need a conceptual stem-cell.
  - Something that can differentiate into huge variety of forms.
  - Find the analogies between forms as aspects within the stem cell.

Category theory as conceptual stem-cell

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- All forms of pure math... (we'll briefly discuss this)
- Databases and knowledge representation (categories and functors)
- Functional programming languages (cartesian closed categories)
- Universal algebra (finite-product categories)
- Dynamical systems and fractals (operad-algebras, co-algebras)
- Hierarchical planning (lenses and monads)
- Shannon Entropy (operad of simplices)
- Partially-ordered sets and metric spaces (enriched categories)
- Higher order logic (toposes = categories of sheaves)
- Measurements of diversity in populations (magnitude of categories)
- Collaborative design (enriched categories and profunctors)
- Petri nets and chemical reaction networks (monoidal categories)
- Quantum processes and NLP (compact closed categories)

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  - Conclude that math/CT explains everything and hence nothing?
- Stem cells don't do work until they differentiate.
  - "Adult-level" work requires differentiation and optimization.
  - But the unified origins lead to impressive interoperability.

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- And it's branched out from math in a big way.
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I propose that CT can really help with information hygiene.

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Let's focus on one.

- Operads: the category-theoretic formalization of "operations".
- It's a branch of category theory that nicely represents the spirit.
- "Operadic": a *theme* of design patterns.

Plan of the talk

I'll discuss CT as mathematics for organizing our thinking.

- Operads, a general framework for compositional operations.
  - I'll sketch a definition and give a lot of examples.
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Let's get into it with *operads*, a framework for compositional operations.

Outline

What are compositional operations?

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Slightly more formal definition to come.

Operads are everywhere

Operads are implicitly being used in many fields.

- Electrical engineering: “wiring diagrams”
- Design: “set-based design”
- Computer programming: “data flow”
- Natural language processing: “grammars”
- Materials science: “hierarchical materials”
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- With mathematical structure, we can go much further.

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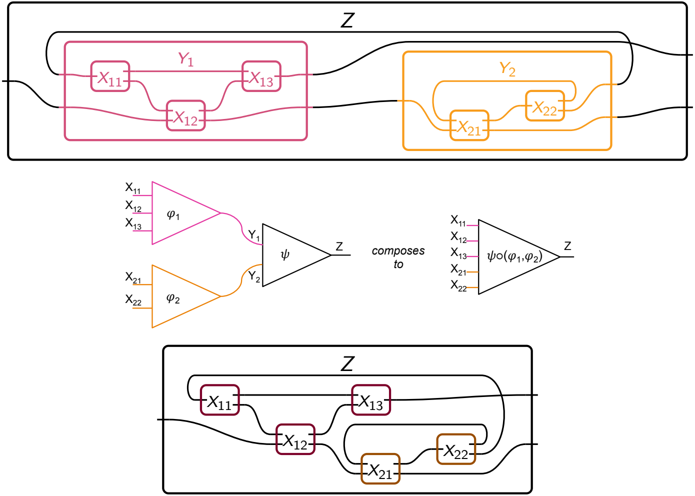
Let's look for sorts, arrangements, and nesting in some examples.

Operad 1: wiring diagrams



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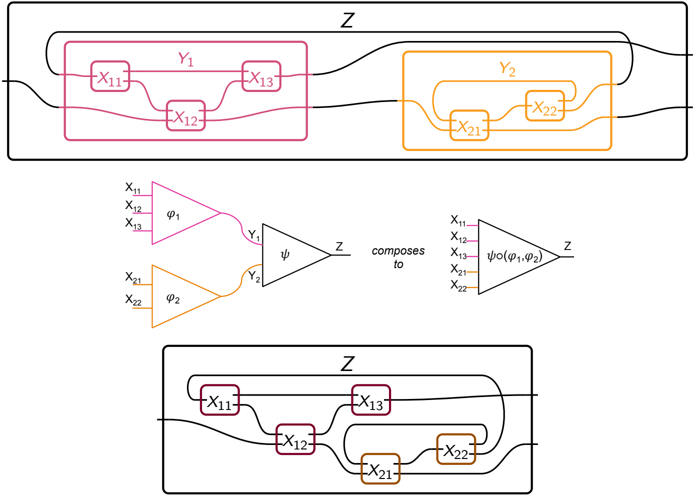

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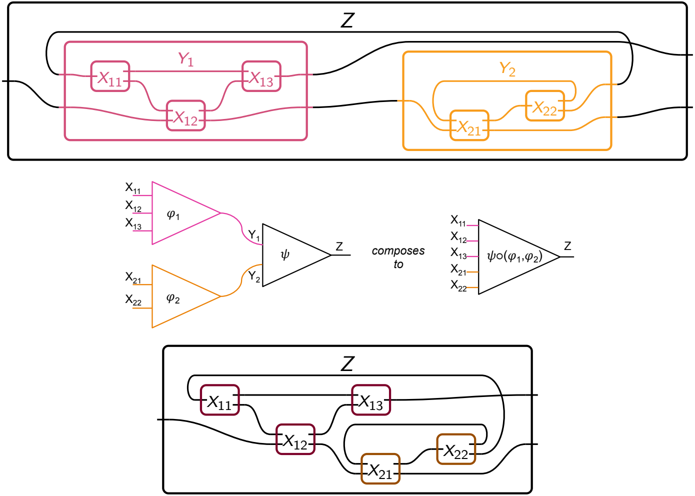


Sorts: boxes with ports. Arrangements: wiring diagrams. Nesting: nesting.

Formal definition of operad

An operad $\mathcal{O}$ consists of

- A set $\mathrm{Ob}(\mathcal{O})$, elements of which are called *sorts*.
- For sorts $X_1, \ldots, X_k, Y \in \mathrm{Ob}(\mathcal{O})$, a set

  $$\mathrm{Mor}_{\mathcal{O}}(X_1, \ldots, X_k; Y)$$

  Its elements are called *morphisms* or arrangements of $X_1, \ldots, X_k$ in Y. A k-ary arrangement $\varphi \in \mathrm{Mor}_{\mathcal{O}}(X_1, \ldots, X_k; Y)$ may be denoted

  $$\varphi \colon (X_1, \ldots, X_k) \to Y.$$

- For each sort $X \in \mathrm{Ob}(\mathcal{O})$, an identity arrangement $\mathrm{id}_X \colon (X) \to X$.
- A composition, or nesting formula, e.g.,

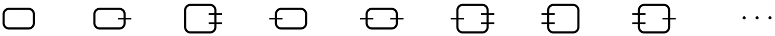
  $$\psi \circ (\varphi_1, \ldots, \varphi_k) \colon (X_{i,j}) \xrightarrow{\varphi_i} (Y_i) \xrightarrow{\psi} Z.$$

These are required to satisfy well-known “unital” and “associative” laws.

Operad 1: WDs again

An operad $\mathcal{W}$ for composing wiring diagrams:

- Sort $X \in \mathcal{W}$: any possible box-with-ports.



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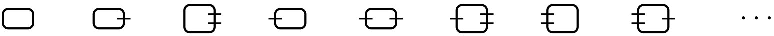
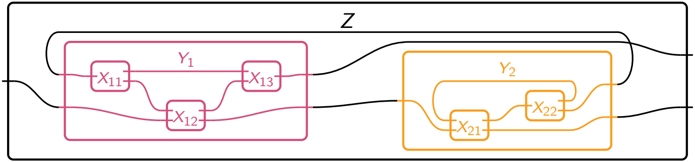


- Arrangement $\varphi \colon X_1, \ldots, X_k \to Y$ in $\mathcal{W}$: any wiring of X's in Y.
- Nesting: the facts about this fractal of wiring possibilities.



$\mathcal{W}$ provides fine-grained control of flow operators and operations.

Operad 2: hierarchical protein materials

There is an operad $\mathcal{M}$ for composing hierarchical protein materials.

- Why protein materials?
  - Protein materials include your skin: stretchable, breathable, waterproof.
  - Eat hamburgers, make amazing material.
  - Materials scientists would *love* to make materials like this.

[1] Giesa, T.; Jagadeesan, R.; Spivak, D.I.; Buehler, M.J. (2015) "Matriarch: a Python library for materials architecture." *ACS Biomaterials Science & Engineering*.

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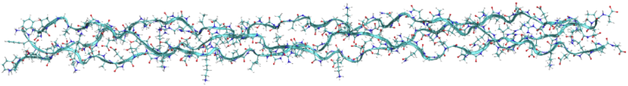
- A protein is an arrangement of simpler proteins.
  - There are "atomic" proteins: amino acids.
  - arrange in series or parallel (H-bonds), or
  - arrange in helices, double helices, any conceivable curve, etc.



- Collagen has a nested structure: it is an array, each fiber of which is a triple helix, each strand of which is a helix, each unit of which is an amino acid.[1]

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Operad 3: probabilities

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Arrangement: "In this event, there's a distribution on next events."

- coin flip: $f = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{P}_2$.
  - In the event coin flip, there's a 50-50 distribution on next events.
- die roll: $r = (\frac{1}{6}, \ldots, \frac{1}{6}) \in \mathcal{P}_6$.
- card selection: $p = (\frac{1}{52}, \ldots, \frac{1}{52}) \in \mathcal{P}_{52}$.

The nesting rule composes distributions by weighted sum:

- Flip a coin: result decides whether to roll a die or pick a card.

$$f \circ (r, p) = \left(\underbrace{\frac{1}{12}, \ldots, \frac{1}{12}}_{\text{6 times}}, \underbrace{\frac{1}{104}, \ldots, \frac{1}{104}}_{\text{52 times}} \right) \in \mathcal{P}_{58}$$

A zoo of operads: Grammars

Any context-free grammar is an operad.

$$
\begin{array}{rcl}
\langle\text{sentence}\rangle & ::= & \langle\text{noun-phrase}\rangle\ \langle\text{verb-phrase}\rangle \\
\langle\text{noun-phrase}\rangle & ::= & \langle\text{pronoun}\rangle \mid \langle\text{proper-noun}\rangle \mid \langle\text{determiner}\rangle\ \langle\text{nominal}\rangle \\
\langle\text{nominal}\rangle & ::= & \langle\text{noun}\rangle \mid \langle\text{nominal}\rangle\ \langle\text{noun}\rangle \\
\langle\text{verb-phrase}\rangle & ::= & \langle\text{verb}\rangle \mid \langle\text{verb}\rangle\ \langle\text{noun-phrase}\rangle \mid \langle\text{verb}\rangle\ \langle\text{prep-phrase}\rangle \\
\langle\text{prep-phrase}\rangle & ::= & \langle\text{preposition}\rangle\ \langle\text{noun-phrase}\rangle
\end{array}
$$

How is this an operad?

- The sorts are the parts of speech.
- The arrangements are the production rules.
- Nesting is nesting.

The operad of sets

There is an operad **Set** of sets and functions.

- The sorts in **Set** are all the sets, e.g. $\{\mathtt{r}, \mathtt{g}, \mathtt{b}\}$, $\mathbb{N}$, $\mathbb{R}^2$, etc.
- An arrangement is a function $f\colon X_1 \times \cdots \times X_k \to Y$.
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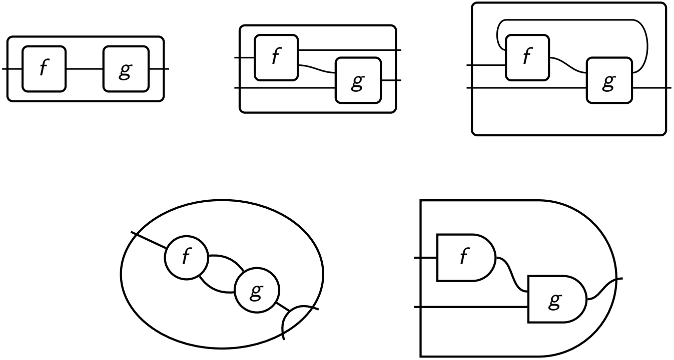
Think of sets as containing features one can sort into.

- An arrangement sorts a k-dimensional feature space into one dimension.

Wiring diagram operads, similar to #1

A representative arrangement $\varphi\colon X, Y \to Z$ in five different WD operads.



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Functors translate between operads

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A functor $F\colon \mathcal{O} \to \mathcal{O}'$ is a translator from one operad to another.

- Functors between grammars are like compilers, or elaborations.
- Functors between wiring diagram operads are sub-languages.
- Functors $\mathcal{O} \to$ **Set** are special: they're called $\mathcal{O}$-*algebras*.

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Algebras *operationalize* operads.

Operads and their algebras

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Operad = theory. Algebras = models.

Each operad has many algebras

Each operad $\mathcal{O}$ is a "theory of composition".

- Sorts $X, Y, \ldots$: what *sorts* of elements in this theory?
- Arrangements φ, ψ: what are the operations?
- Nesting: what kind of laws?

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The $\mathcal{O}$-algebras $A\colon \mathcal{O} \to \mathbf{Set}$ are the models of theory $\mathcal{O}$.

- An $\mathcal{O}$-algebra A says what's actually being composed.
  - To each sort X: a set $A(X)$ of elements.
  - To each arrangement φ: a k-ary operation $A(\varphi)$.
    - If $\varphi\colon X_1, \ldots, X_k \to Y$ is an arrangement,
    - Then $A(\varphi)\colon A(X_1) \times \cdots \times A(X_k) \to A(Y)$ is a function.
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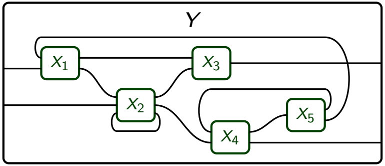
Next, I'll explain what algebras on an operad look like.

Multiple algebras for wiring diagrams operad

Recall operad $\mathcal{W}$ of all boxes X and WDs, $\varphi\colon X_1, \ldots, X_5 \to Y$.[2]



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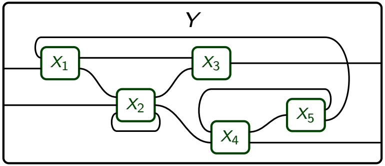

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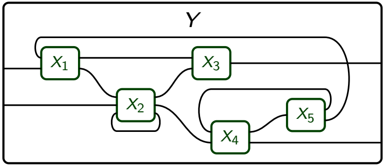

There are many different $\mathcal{W}$-algebras $A\colon \mathcal{W} \to \mathbf{Set}$.

- Open dynamical systems: machines with ports.
  - $A(X)$: the set of all DS's with input-output shape X.
  - $A(\varphi)$: a certain variable-sharing formula.

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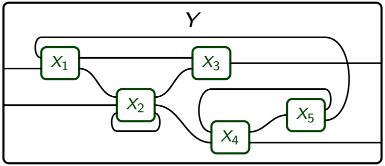

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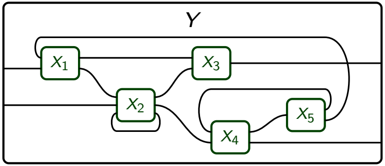
[2] Imagine each port / wire labeled by a topological space: the signal space.

Multiple algebras for wiring diagrams operad

Recall operad $\mathcal{W}$ of all boxes X and WDs, $\varphi\colon X_1,\dots,X_5 \to Y$.[2]



There are many different $\mathcal{W}$-algebras $A\colon \mathcal{W} \to \mathbf{Set}$.

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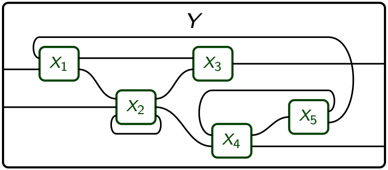
$\mathcal{W}$-algebra of tensor networks

We said $\mathcal{W}$ is modeled by dynamical systems.



Another algebra $\mathcal{W} \to \mathbf{Set}$: tensors (not dynamic at all).[3]

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- Wiring diagram = tensor network.
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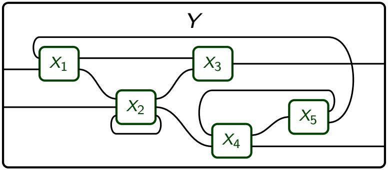

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Later: a relationship between dynamical systems and tensors...

- ... which led me to a numerical method for solving systems.
- The pixel array method, up next.

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Outline

Detailed look at a real-world application

Separately plot the solutions to equations: $f(x, w) = 0$ and $g(w, y) = 0$.

- Plot each in a bounding box, e.g. $[-1.5, 1.5]$.
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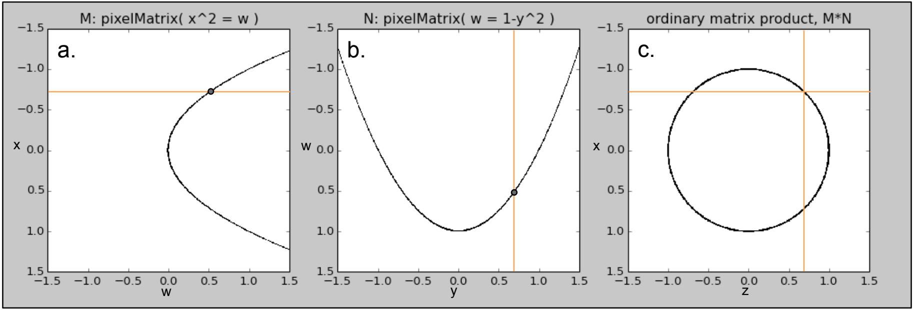
Example: $x^2 = w$ and $w = 1 - y^2$.



A more complex example

The following eq's are not differentiable, nor even defined everywhere.

$$\cos\left(\ln(z^2 + 10^{-3}x)\right) - x + 10^{-5}z^{-1} = 0 \qquad \text{(Equation 1)}$$

$$\cosh(w + 10^{-3}y) + y + 10^{-4}w = 2 \qquad \text{(Equation 2)}$$

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Q: For what values of w and z does a simultaneous solution exist? [4]

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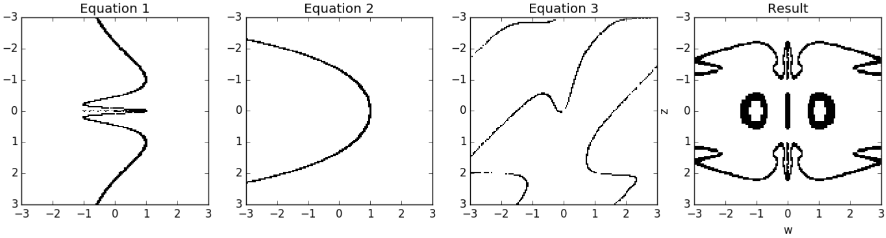

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Equations and wiring diagrams

Consider an arbitrary system of equations having the following form:

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Bold variables are those we want to *expose*; others are *unexposed*.

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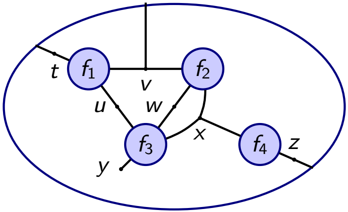
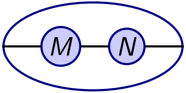


Said another way, we want $\{(t, v, z) \mid \exists u, w, x, y : f_1 = f_2 = f_3 = f_4 = 0\}$.

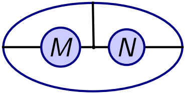
Example wiring diagrams for named operations

Some famous matrix products as wiring diagrams:

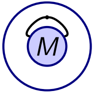
Multiplication: MN



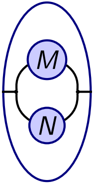
Khatri-Rao: $M \odot N$



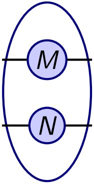
Trace: $\mathrm{Tr}(M)$



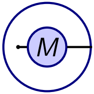
Hadamard: $M \circ N$



Kronecker: $M \otimes N$



Marginalize: $\sum_i M_{i,j}$



Speed and accuracy

- The PA method has good accuracy guarantees.
  - No false negatives, only false positives.
  - As pixel density $\to \infty$, false positive $\to 0$
  - Rate is proportional to size of largest derivative.
- The PA method is faster than Newton-style methods....
  - If you want all solutions in a bounding box...
- Applications: plot steady states of dynamical systems.

PA method for dynamical systems

"We can use the PA method to find the steady states of dynamic systems."

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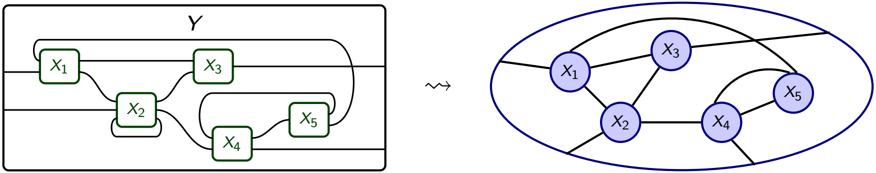

And we want to translate dynamical systems to pixel arrays.

$$\begin{tikzcd}
\mathcal{W}_{\text{box}} \arrow[rr, "\text{as shown above}"] \arrow[dr, "\text{DynSys}"'] & ? & \mathcal{W}_{\text{circle}} \arrow[dl, "\text{PixArray}"] \\
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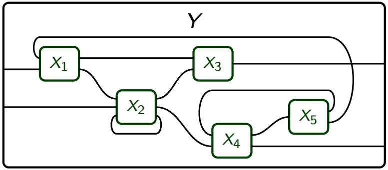
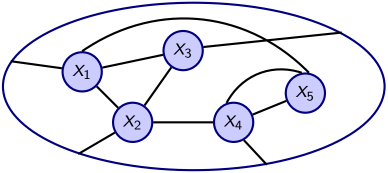


$\rightsquigarrow$



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- Power network model: a binary tree of nodes ("buses")
    - Each node i has four associated variables: P_i, Q_i, V_i, θ_i.
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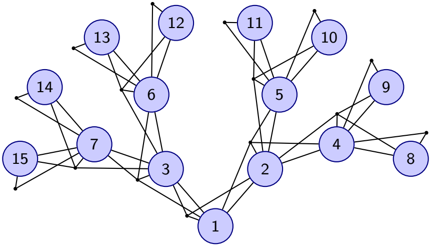
- This is perfect for the PA method.



- Given parallel resources, runtime is logarithmic in # nodes.

Outline

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- We briefly discussed maps between functors.
  - Steady states of dynamical systems arranged as tensors.
  - (Another example: discretization of cont's dynamical systems.)

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The point of all that was to give a glimpse into category theory.

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There's even a category of categories and an operad of operads.

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Thanks! Questions and comments welcome.