

Session 10 - Parallelism

Applied Compositional Thinking for Engineers

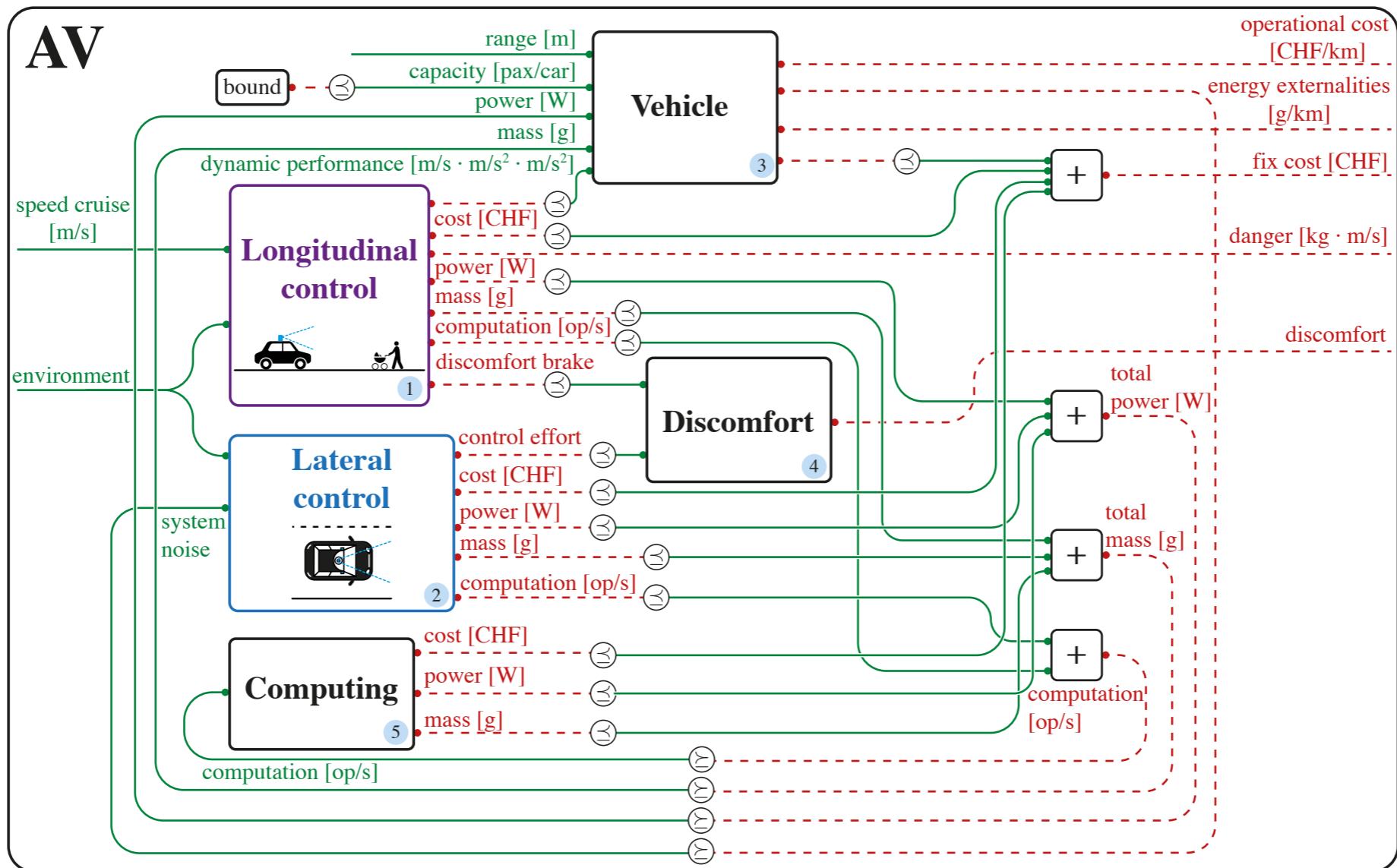


Logistics, announcements

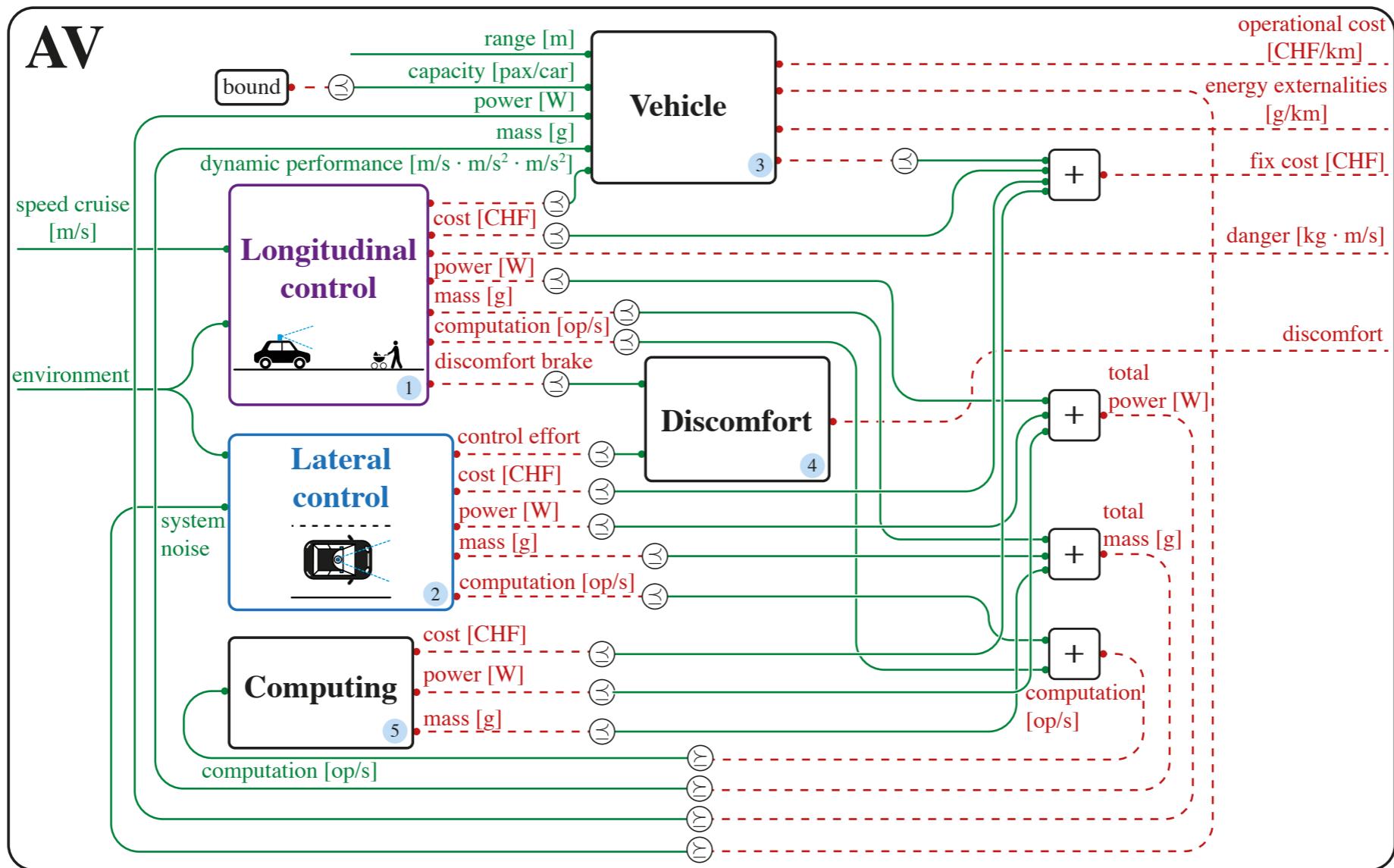
This weekend:

- ▶ **Saturday, January 30th at 09:00 UTC:** Office hour;

Today: Parallelism



Today: Parallelism



Today:

- ▶ Monoids, monoidal posets, wiring diagrams;
- ▶ Monoidal categories;
- ▶ Enrichments;
- ▶ Locally posetal structures.

Monoid

- ▶ A **monoid** consists of:
 1. A set M ;
 2. A *neutral element* $e \in M$;
 3. An operation $(\ast) : M \times M \rightarrow M$;
- ▶ These satisfy:
 - a) Associative law: $(x \ast y) \ast z = x \ast (y \ast z)$;
 - b) Unit Laws: $1 \ast x = x = x \ast 1$.

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a) Associative law: $(x * y) * z = x * (y * z)$;

b) Unit Laws: $\underset{e}{\cancel{x}} * x = x = x * \underset{e}{\cancel{x}}$.

► $\langle \mathbb{R}, +, 0 \rangle$:
$$\begin{aligned} (x+y)+z &= x+(y+z) & , \quad x, y, z \in \mathbb{R} \\ 0+x &= x = x+0 \end{aligned}$$

► $\langle \mathbb{R}_{\geq 0}, \max, 0 \rangle$:
$$\begin{aligned} \max(\overbrace{\max(x, y)}, z) &= \max(x, y, z) \\ &= \max(x, \underbrace{\max(y, z)}) \\ \max(0, x) &= x = \max(x, 0) \end{aligned}$$

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$$(x + y) + z = x + (y + z)$$

and

$$0 + x = x = x + 0$$

► $\langle \mathbb{R}_{\geq 0}, \max, 0 \rangle$:

$$\begin{aligned} \max(\max(x, y), z) &= \max(x, y, z) \\ &= \max(x, \max(y, z)) \end{aligned}$$

and

$$\max(0, x) = x = \max(x, 0)$$

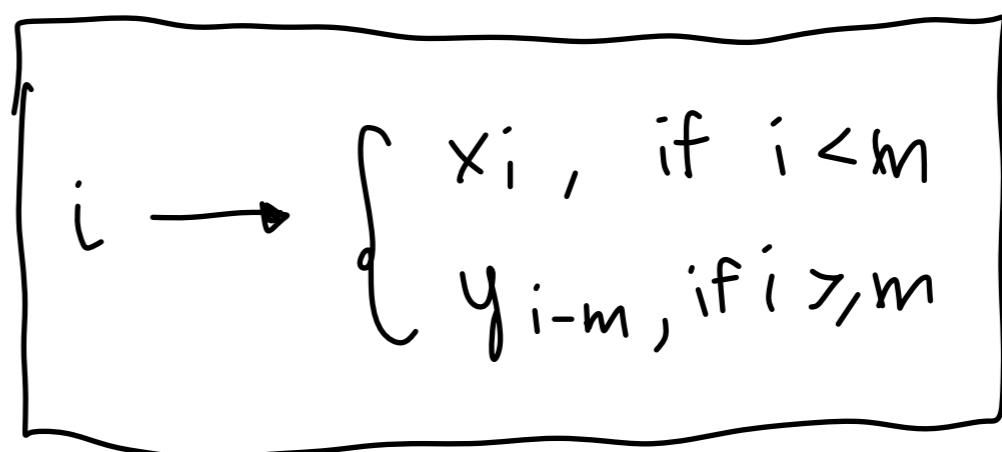
Lists as monoids

- ▶ A **sequence** is a function $f : A \rightarrow S$, with $A \subseteq \mathbb{N}_0$;
- ▶ A sequence is **finite** if A is finite;
- ▶ Finite sequences are often called **lists**;
- ▶ Given S , denote the set of all lists on S by S^* .

• $s \in S^*$, $n \in \mathbb{N}_0$, $f : [n] \rightarrow S$ with $[n] = \{i \in \mathbb{N} \mid i \leq n\}$

$0, \dots, n-1$ to s_0, \dots, s_{n-1}

• $x \in S^*$ of length m
 $y \in S^*$ " " n
concatenation :



↳ free monoid

Lists as monoids

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- ▶ Finite sequences are often called **lists**;
- ▶ Given S , denote the set of all lists on S by S^* .

A list is a $s \in S^*$ and consists of a $n \in \mathbb{N}_0$ and $f : [n] \rightarrow S$, with $[n] = \{i : \mathbb{N} \mid i < n\}$.

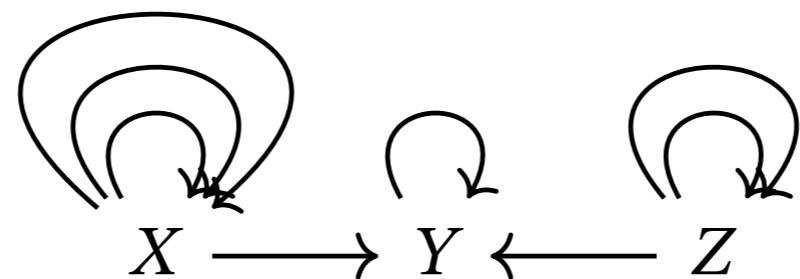
It assigns $0, \dots, n - 1$ to s_0, \dots, s_{n-1} ;

Empty list is the unique list of length 0;

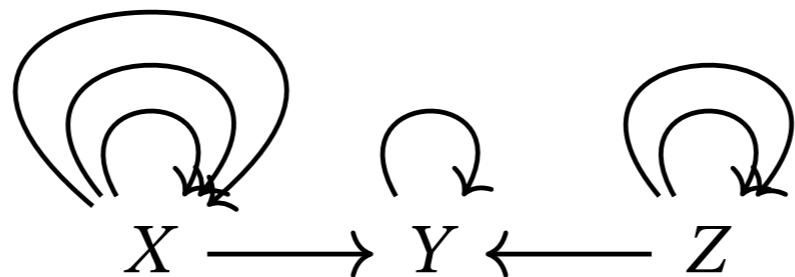
Given a list $x \in S^*$ of length m and a list $y \in S^*$ of length n , we can define their *concatenation* $x * y$ as list of length $m + n$ with:

$$i \mapsto \begin{cases} x_i & \text{if } i < m \\ y_{i-m} & \text{if } i \geq m. \end{cases}$$

Endomorphisms in every category form a monoid



Endomorphisms in every category form a monoid



- ▶ For any category \mathbf{C} , and any $X \in \text{Ob}_{\mathbf{C}}$, $\text{Hom}_{\mathbf{C}}(X, X)$ is a monoid:
 - $*$: composition “ \circ ” in \mathbf{C} ;
 - e : identity id_X ;
- ▶ Associativity and unitality follow from \mathbf{C} ;
- ▶ A monoid is a **one-object** category.

Monoidal poset definition

- A (***symmetric***) ***monoidal structure*** on a poset $\langle P, \leq \rangle$ consists of:
 1. An element $I \in P$, called **monoidal unit**, and
 2. a function $\otimes : P \times P \rightarrow P$, called the **monoidal product**. Note that we write

$$\otimes(p_1, p_2) = p_1 \otimes p_2, \quad p_1, p_2 \in P.$$

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- The constituents must satisfy:
 - a. *Monotonicity*: For all $p_1, p_2, q_1, q_2 \in P$, if $p_1 \leq q_1$ and $p_2 \leq q_2$, then

$$p_1 \otimes p_2 \leq q_1 \otimes q_2.$$

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- b. *Unitality*: For all $p \in P$: $I \otimes p = p$ and $p \otimes I = p$.

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 - c. *Associativity*: For all $p, q, r \in P$: $(p \otimes q) \otimes r = p \otimes (q \otimes r)$.

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 - b. *Unitality*: For all $p \in P$: $I \otimes p = p$ and $p \otimes I = p$.
 - c. *Associativity*: For all $p, q, r \in P$: $(p \otimes q) \otimes r = p \otimes (q \otimes r)$.
 - d. *Symmetry*: For all $p, q \in P$: $p \otimes q = q \otimes p$.
- A poset equipped with $\langle P, \leq, I, \otimes \rangle$ is called a **monoidal poset**.

Monoidal poset examples

► $\langle \mathbb{R}, \leq, 0, \oplus \rangle$:

- $x_1 \leq y_1, x_2 \leq y_2 \Rightarrow x_1 + x_2 \leq y_1 + y_2$
- $0 + x = x = x + 0$
- $(x+y)+z = x+(y+z)$
- $x+y = y+x$

► $\langle \mathbb{R}, \leq, 0, \otimes \rangle$:

$$-5 \leq 0, -4 \leq 3 \Rightarrow (-5) \cdot (-4) \neq 0 \cdot 3$$

$$20 \neq 0$$

► $\langle \text{Bool}, \leq_{\text{Bool}}, \text{true}, \wedge \rangle$:

| | |
|-------|--|
| true | |
| | |
| false | |

| | | |
|-------|-------|-------|
| 1 | false | true |
| | false | false |
| false | true | false |

- $x_1 \leq y_1, x_2 \leq y_2 \Rightarrow x_1 \wedge x_2 \leq y_1 \wedge y_2$

- $x \wedge \text{true} = x = \text{true} \wedge x$

- $x \wedge y = y \wedge x$

Monoidal poset examples

► $\langle \mathbb{R}, \leq, 0, + \rangle$:

- Given $x_1 \leq y_1, x_2 \leq y_2$: $x_1 + x_2 \leq y_1 + y_2$;
- $0 + x = x = x + 0$;
- $(x + y) + z = x + (y + z)$;
- $x + y = y + x$.

► $\langle \mathbb{R}, \leq, 1, * \rangle$:

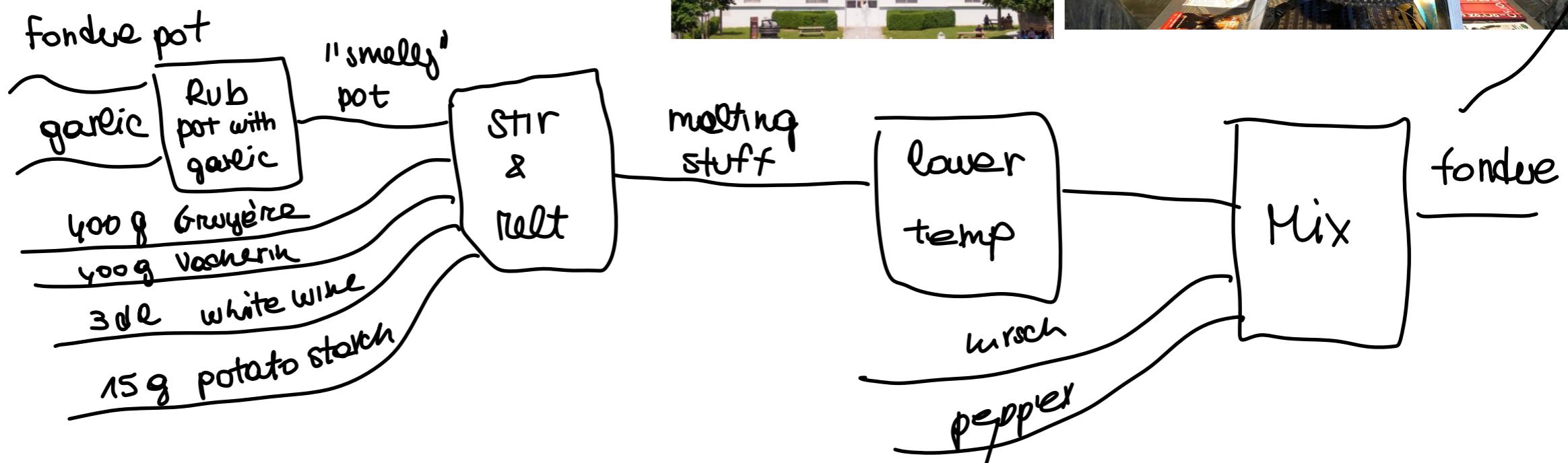
- $-5 \leq 0$ and $-4 \leq 3$;
- $-5 \cdot -4 \not\leq 3 \cdot 0$

► $\langle \text{Bool}, \leq_{\text{Bool}}, \text{true}, \wedge \rangle$:

| \wedge | false | true |
|----------|-------|-------|
| false | false | false |
| true | false | true |

- $x \wedge \text{true} = x = \text{true} \wedge x$;
- $x_1 \leq y_1, x_2 \leq y_2$: $x_1 \wedge x_2 \leq y_1 \wedge y_2$.

(Swiss) wiring diagrams

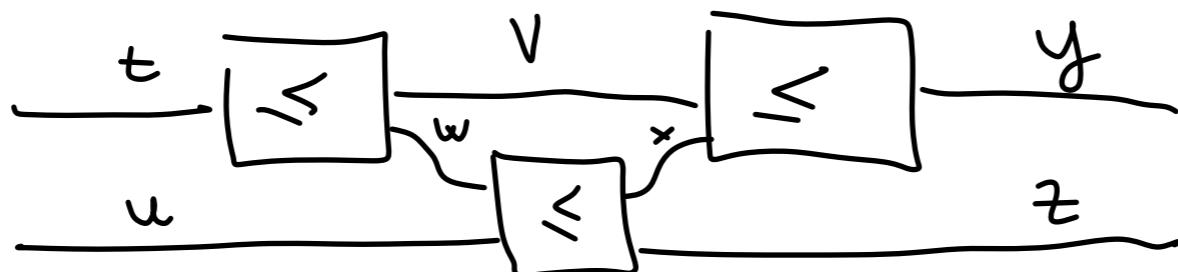


Introducing wiring diagrams

- ▶ “Visual representations for building new **relationships** from old”;
- ▶ In monoidal posets we have “ \leq ” (series):



- ▶ In symmetric monoidal posets, we can also do parallel:



- ▶ Monoidal product:

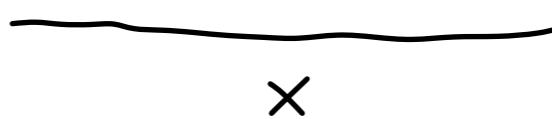


- ▶ Monoidal unit:

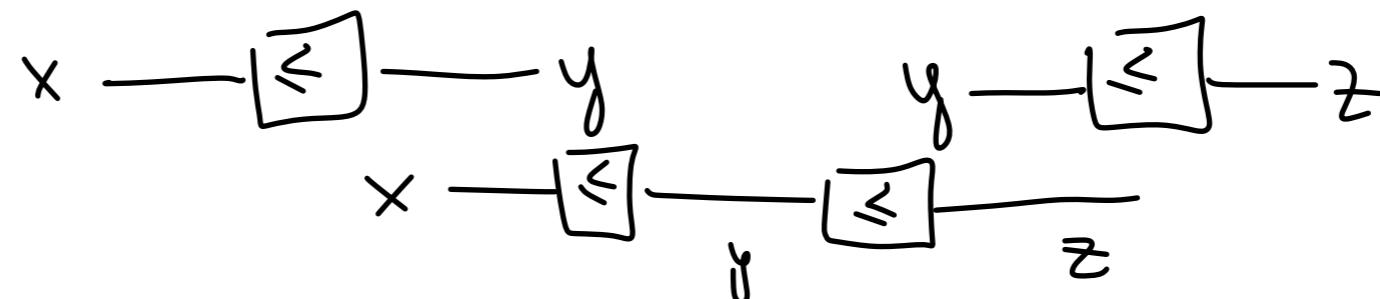


Wiring diagrams for symmetric monoidal posets

► Reflexivity:



► Transitivity:



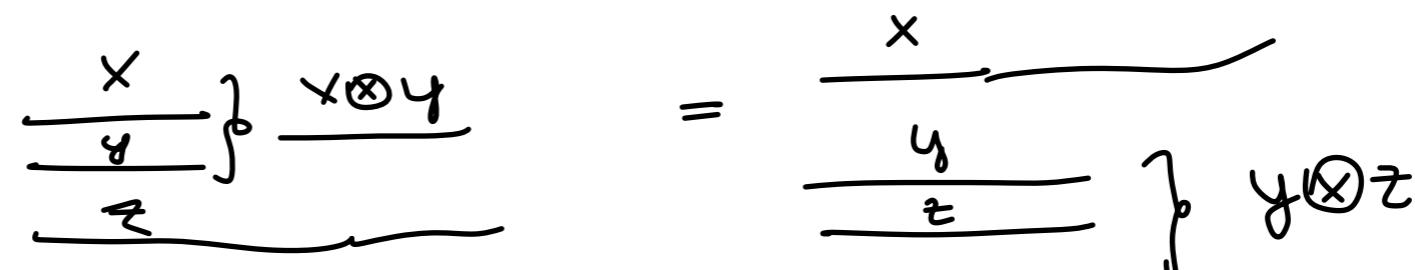
► Monotonicity:



► Unitality:



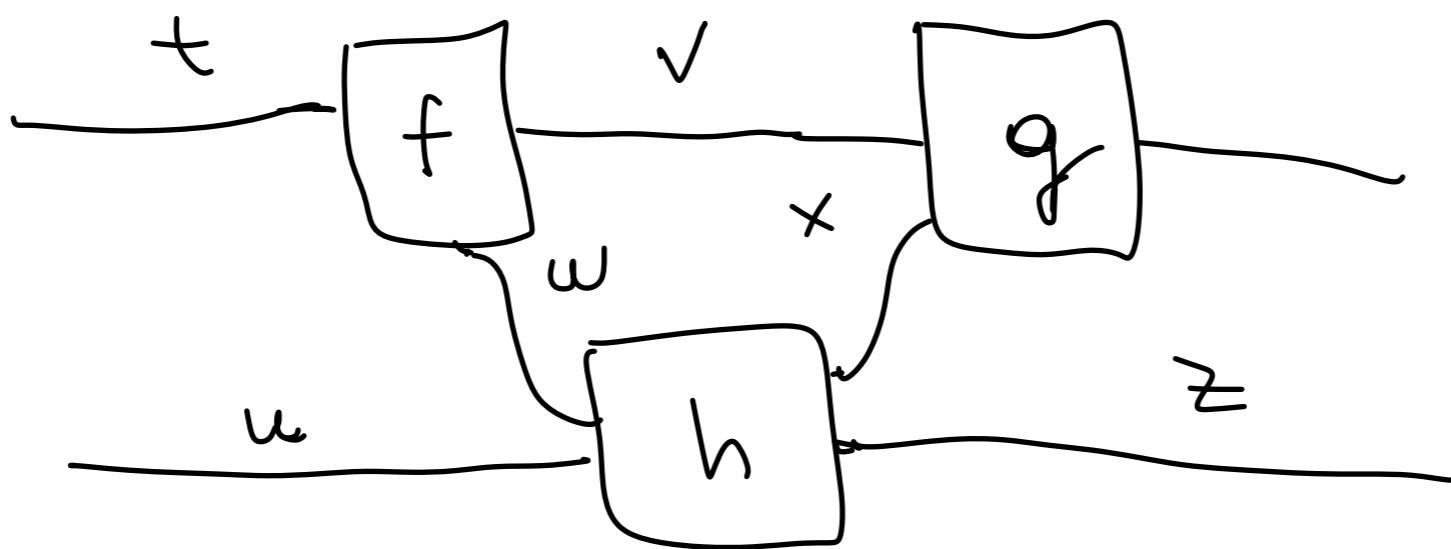
► Associativity:



► Symmetry:



From monoidal posets to monoidal categories



Definition of monoidal category

- ▶ A **monoidal structure** on a category \mathbf{C} consists of:
 1. An object $I \in \text{Ob}_{\mathbf{C}}$ called the **monoidal unit**;
 2. A functor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, called the **monoidal product**.

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- ▶ These are subject to natural isomorphisms:
 3. $\lambda_c : I \otimes c \cong c$ for every $c \in \text{Ob}_{\mathbf{C}}$,
 4. $\rho_c : c \otimes I \cong c$ for every $c \in \text{Ob}_{\mathbf{C}}$,
 5. $\alpha_{c,d,e} : (c \otimes d) \otimes e \cong c \otimes (d \otimes e)$ for every $c, d, e \in \text{Ob}_{\mathbf{C}}$.

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 5. $\alpha_{c,d,e} : (c \otimes d) \otimes e \cong c \otimes (d \otimes e)$ for every $c, d, e \in \text{Ob}_{\mathbf{C}}$.
- ▶ These are required to satisfy:
 - The **triangle** identity:

$$\begin{array}{ccc} (c \otimes I) \otimes d & \xrightarrow{\alpha_{c,I,d}} & c \otimes (I \otimes d) \\ \rho_c \otimes I \searrow & & \swarrow I \otimes \lambda_d \\ & c \otimes d & \end{array}$$

- The **pentagon** identity:

$$\begin{array}{ccccc} & & (a \otimes b) \otimes (c \otimes d) & & \\ & \nearrow \alpha_{a \otimes b,c,d} & & \searrow \alpha_{a,b,c \otimes d} & \\ ((a \otimes b) \otimes c) \otimes d & & & & (a \otimes (b \otimes (c \otimes d))) \\ \alpha_{a,b,c} \otimes \text{id}_d \downarrow & & & & \uparrow \text{id}_a \otimes \alpha_{b,c,d} \\ (a \otimes (b \otimes c)) \otimes d & \xrightarrow{\alpha_{a,b \otimes c,d}} & & & a \otimes ((b \otimes c) \otimes d) \end{array}$$

- ▶ A category equipped with a monoidal structure is called a **monoidal category**.

Monoidal structure on Set

- $\langle \text{Set}, \times, \{\ast\} \rangle$, with $A, B \in \text{Ob}_{\text{Set}}$
- $A \times B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$
- $f: A \rightarrow A'$, $g: B \rightarrow B'$

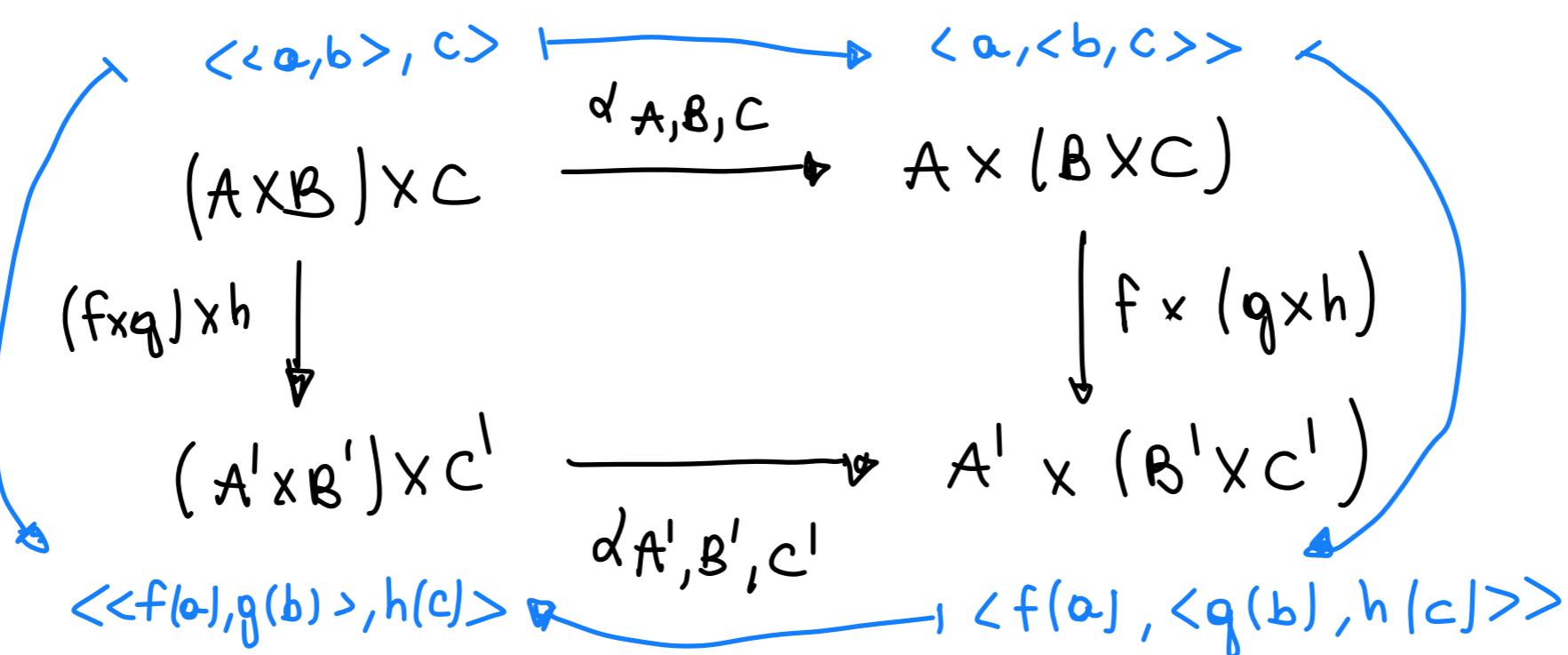
$$(f \times g): A \times B \longrightarrow A' \times B'$$

$$\langle a, b \rangle \longmapsto \langle f(a), g(b) \rangle$$
- $\alpha_{A, B, C}: (A \times B) \times C \xrightarrow{\sim} A \times (B \times C)$

$$\langle \langle a, b \rangle, c \rangle \longmapsto \langle a, \langle b, c \rangle \rangle$$

↳ isomorphism

↳ natural?



Monoidal structure on Set

- $\lambda_A : \{\star\} \times A \xrightarrow{\sim} A$
 $\langle \star, a \rangle \mapsto a$

again, I can go back
 ↳ isomorphism.

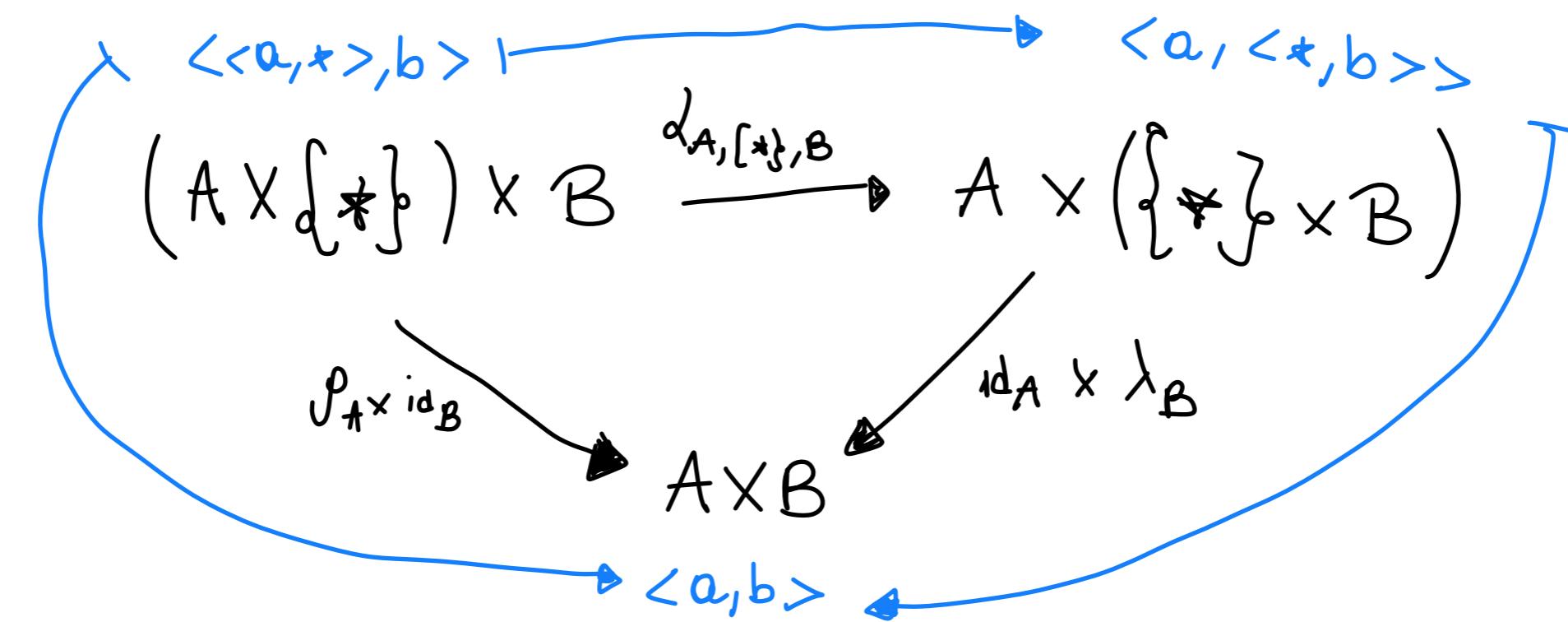
- $f: A \rightarrow A'$

- $\rho_A: A \times \{\star\} \xrightarrow{\sim} A$
 $\langle a, \star \rangle \mapsto a$

Monoidal structure on Set

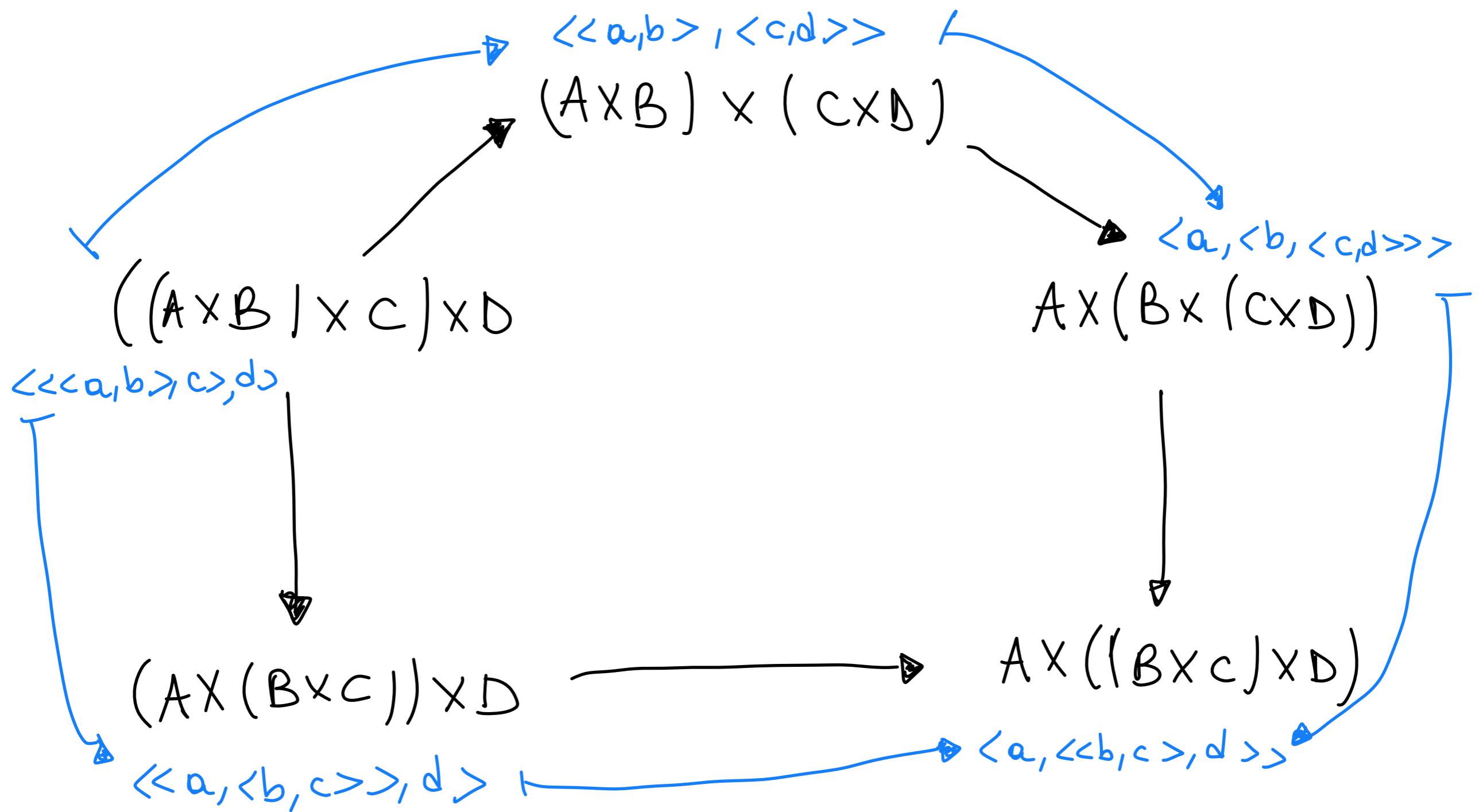
Triangle:

$A, B \in \text{Ob}_{\text{Set}}$



Monoidal structure on Set

Pentagon Identity:



Definition of symmetric monoidal category

- Take a monoidal category $\langle \mathbf{C}, \otimes, I \rangle$, consider $c, d \in \text{Ob}_{\mathbf{C}}$;

Definition of symmetric monoidal category

- ▶ Take a monoidal category $\langle \mathbf{C}, \otimes, I \rangle$, consider $c, d \in \text{Ob}_{\mathbf{C}}$;
- ▶ A **symmetric structure** on it consists of an isomorphism $\sigma_{c,d} : (c \otimes d) \xrightarrow{\cong} (d \otimes c)$, called the *braiding*. The braiding must satisfy, $\forall c, d \in \text{Ob}_{\mathbf{C}}$:
 1. *Naturality*: Given any morphisms $f_1 : c_1 \rightarrow d_1$ and $f_2 : c_2 \rightarrow d_2$:

$$\begin{array}{ccc} c_1 \otimes c_2 & \xrightarrow{f_1 \otimes f_2} & d_1 \otimes d_2 \\ \sigma \downarrow & & \downarrow \sigma \\ c_2 \otimes c_1 & \xrightarrow{f_2 \otimes f_1} & d_2 \otimes d_1 \end{array}$$

2. Given any $c, d \in \text{Ob}_{\mathbf{C}}$:

$$\begin{array}{ccc} I \otimes c & \xrightarrow{\sigma} & c \otimes I \\ & \searrow \lambda \quad \swarrow \rho & \\ & c & \end{array} \qquad \begin{array}{ccc} c \otimes d & \xrightarrow{\sigma} & d \otimes c \\ & \equiv & \\ & c \otimes d & \swarrow \sigma \end{array}$$

3. *Hexagon identity*: Given any $c, d, e \in \text{Ob}_{\mathbf{C}}$:

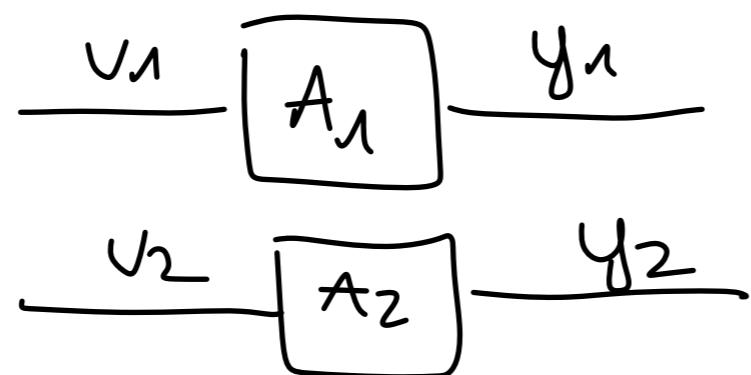
$$\begin{array}{ccccc} (c \otimes d) \otimes e & \xrightarrow{\sigma_{c,d} \otimes \text{id}_e} & (d \otimes c) \otimes e & \xrightarrow{\alpha_{d,c,e}} & d \otimes (c \otimes e) \\ \alpha_{c,d,e} \downarrow & & & & \downarrow \text{id}_d \otimes \sigma_{c,e} \\ c \otimes (d \otimes e) & \xrightarrow{\sigma_{c,d \otimes e}} & (d \otimes e) \otimes c & \xrightarrow{\alpha_{d,e,c}} & d \otimes (e \otimes c) \end{array}$$

Other examples of monoidal categories

- ▶ **Exercise:** Check that $\langle \mathbf{Set}, \times, \{\ast\} \rangle$ forms a symmetric monoidal category.

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- ▶ $\langle \mathbf{Set}, \cup, \emptyset \rangle$;
- ▶ $\langle \mathcal{P}(A), \cap, A \rangle$ and $\langle \mathcal{P}(A), \cup, \emptyset \rangle$;
- ▶ $\langle \mathbf{Vect}, \otimes, 1 \rangle$;
- ▶ $\langle \mathbf{Vect}, \oplus, 0 \rangle$:



$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_2 \end{pmatrix}$$

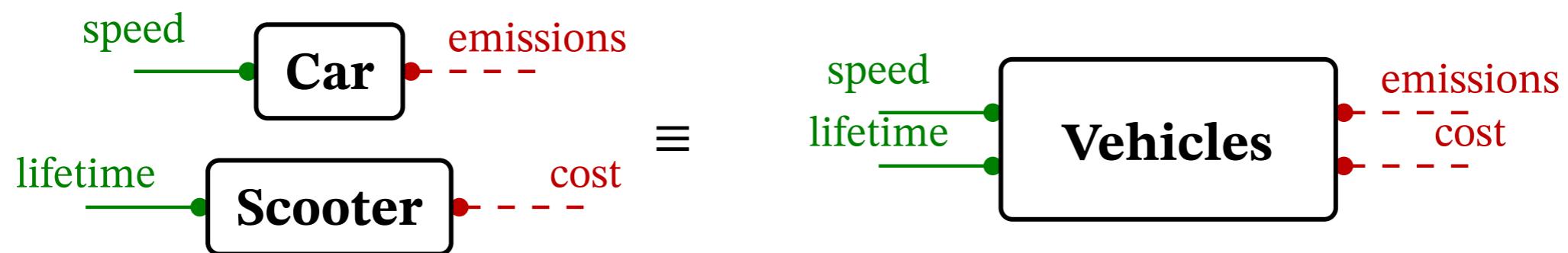
Other examples of monoidal categories

- ▶ **Exercise:** Check that $\langle \mathbf{Set}, \times, \{\ast\} \rangle$ forms a symmetric monoidal category.
- ▶ $\langle \mathbf{Set}, \cup, \emptyset \rangle$;
- ▶ $\langle \mathcal{P}(A), \cap, A \rangle$ and $\langle \mathcal{P}(A), \cup, \emptyset \rangle$;
- ▶ $\langle \mathbf{Vect}, \otimes, 1 \rangle$;
- ▶ $\langle \mathbf{Vect}, \oplus, 0 \rangle$:

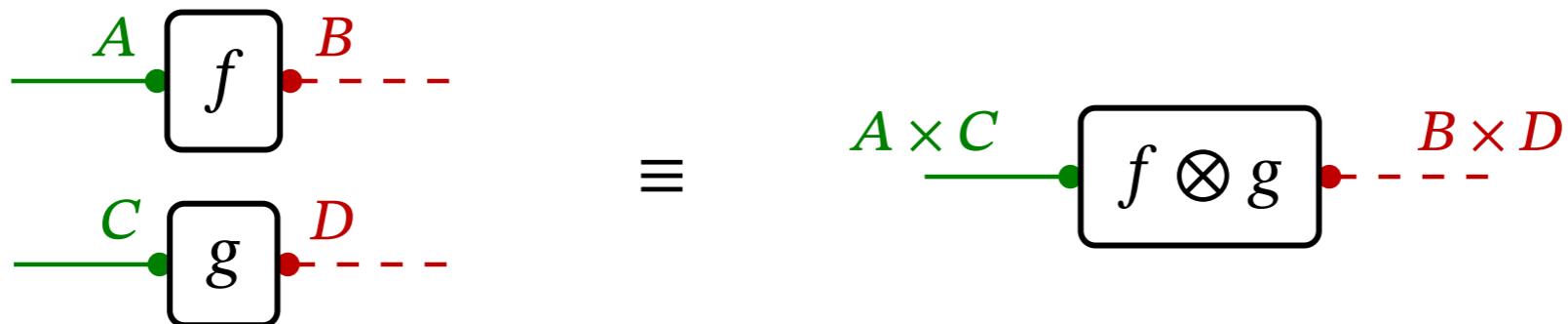
- ▶ When the monoidal product is the categorical product: **cartesian monoidal category**.

Semantics of monoidal in DP

- ▶ Two design problems in **parallel**;



DP is (symmetric) monoidal



- ▶ Take design problems $f : A \rightarrow B$, $g : C \rightarrow D$;
- ▶ The **monoidal unit** is a singleton $\{*\}$;
- ▶ Their **monoidal product** is:
$$f \otimes g : (A \times C)^{\text{op}} \times (B \times D) \rightarrow_{\text{Pos}} \mathbf{Bool},$$
$$\langle \langle a, c \rangle^*, \langle b, d \rangle \rangle \mapsto f(a^*, b) \wedge g(c^*, d).$$
- ▶ This (and some more things) form a symmetric monoidal category.

Enriched categories

- ▶ Given a category \mathbf{C} , $\text{Hom}_{\mathbf{C}}$ is usually a **set**;
- ▶ What if this has a more specific structure?
 - Topological space;
 - Vector space;
 - Poset.

- ▶ Let $\langle \mathbf{V}, \otimes, I \rangle$ be a monoidal category ;
- ▶ A category \mathbf{C} **enriched** in \mathbf{V} is composed of:
 1. Objects: A set $\text{Ob}_{\mathbf{C}}$ of objects;
 2. Objects of morphisms: For each $x, y \in \text{Ob}_{\mathbf{C}}$, an hom-object $\text{Hom}_{\mathbf{C}}(x, y) \in \text{Ob}_{\mathbf{V}}$
 3. Composition morphism: for each $x, y, z \in \text{Ob}_{\mathbf{C}}$, a morphism
$$\circ : \text{Hom}_{\mathbf{C}}(x, y) \otimes \text{Hom}_{\mathbf{C}}(y, z) \rightarrow \text{Hom}_{\mathbf{C}}(x, z)$$
 4. Identity element: For any $x \in \text{Ob}_{\mathbf{C}}$, an identity element $\text{id}_x : I \rightarrow \text{Hom}_{\mathbf{C}}(x, x)$.

A category enriched in **Bool** is a preorder

- Let's take the category **Bool** (with $\text{false} \leq \text{true}$)
 - $\langle \text{Bool}, \wedge, \text{true} \rangle$ is monoidal
 - $\text{Ob}_{\text{Bool}} = \{\text{false}, \text{true}\}$
 - Take any category C , enrich in $\langle \text{Bool}, \wedge, \text{true} \rangle$:
 - ↳ Objects: Ob_C
 - ↳ For each $x, y \in \text{Ob}_C$: $\text{Hom}_C(x, y) \in \text{Ob}_{\text{Bool}}$
 - ↳ assign each morphism to either true or false
 - ↳ If $x, y, z \in \text{Ob}_C$: ; : $\text{Hom}_C(x, y) \wedge \text{Hom}_C(y, z) \xrightarrow{\quad} \text{Hom}_C(x, z)$
 - ↳ $x \leq y \wedge y \leq z \Rightarrow x \leq z$ (transitivity)
 - ↳ id_x : $\text{true} \leq \text{Hom}_C(x, x)$
 - $x \leq x$ (reflexivity)
 - ⇒ if $f \in \text{Hom}_C(x, y)$ is true, $x \sim y$, else $x \not\sim y \Rightarrow \underline{\text{preorder}}$

Enriched in Pos: locally posetal

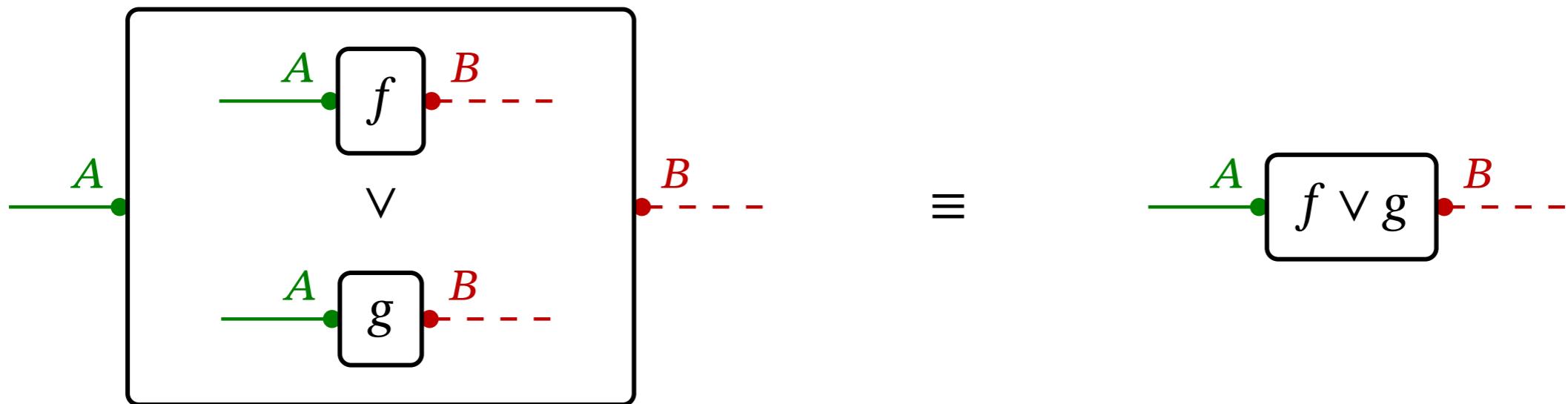
- ▶ A 2-category \mathbf{C} is **locally posetal** if it is **Pos**-enriched;
- ▶ For $x, y \in \text{Ob}_{\mathbf{C}}$: $\text{Hom}_{\mathbf{C}}(x, y)$ is a poset

Enriched in Pos: locally posetal

- ▶ A 2-category **C** is **locally posetal** if it is **Pos**-enriched;
 - ▶ For $x, y \in \text{Ob}_C$: $\text{Hom}_C(x, y)$ is a poset
-
- ▶ **DP** is locally posetal;
 - ▶ Morphisms in **DP** are boolean profunctors;
 - ▶ Given $f, g : \textcolor{green}{A} \rightarrow \textcolor{red}{B}$, we say “**f implies g**”, denoted $f \leq_{\text{DP}} g$, if

$$f(\textcolor{green}{a}^*, \textcolor{red}{b}) \leq_{\text{Bool}} g(\textcolor{green}{a}^*, \textcolor{red}{b}) \quad \forall \textcolor{green}{a} \in \textcolor{green}{A}, \textcolor{red}{b} \in \textcolor{red}{B}.$$

Join in DP



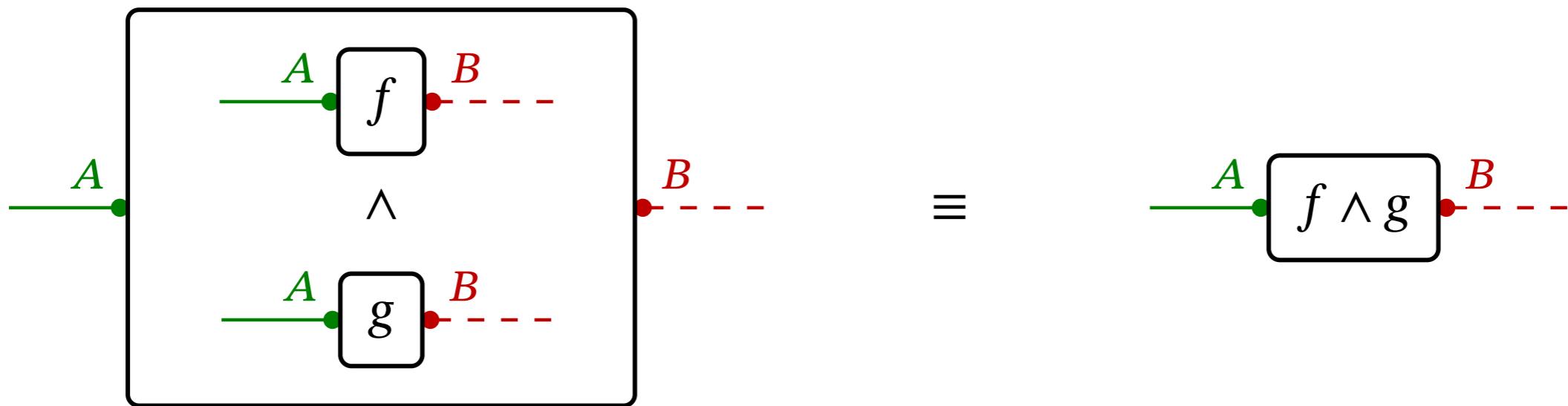
- ▶ Consider $f, g : A \rightarrow B$;
- ▶ Their **join** is:

$$(f \vee g) : A^{\text{op}} \times B \rightarrow_{\text{Pos}} \text{Bool}$$

$$\langle a^*, b \rangle \mapsto f(a^*, b) \vee g(a^*, b).$$

- ▶ “Choose any”: interchangeable technologies.

Meet in DP



► Consider $f, g : A \rightarrow B$;

► Their **meet** is:

$$(f \wedge g) : A^{\text{op}} \times B \rightarrow_{\mathbf{Pos}} \text{Bool}$$

$$\langle a^*, b \rangle \mapsto f(a^*, b) \wedge g(a^*, b).$$

► “Convince two experts”;

► Specific case of monoidal product

DP is locally latticeal

- ▶ DP is enriched in **BoundedLat**;

DP is locally latticeal

- ▶ DP is enriched in **BoundedLat**;
- ▶ Consider $f, g : A \rightarrow B$:

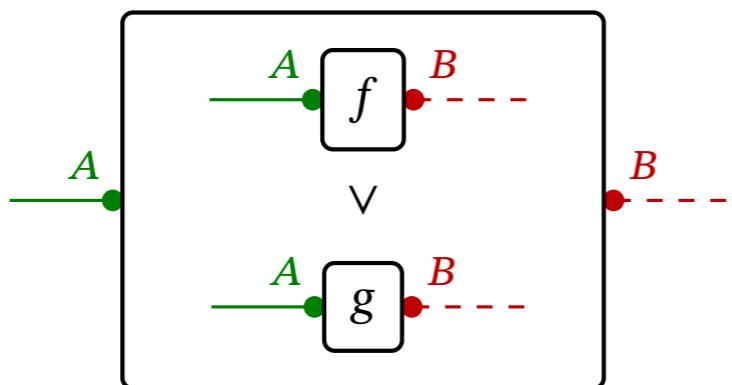
$\perp_{A,B}$

$f \wedge g$

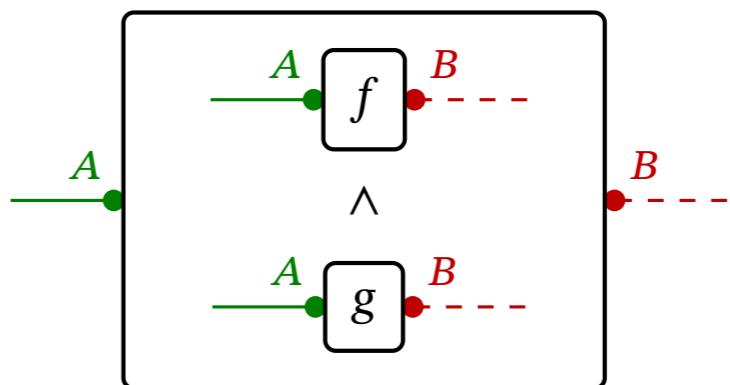
$f \vee g$

$\top_{A,B}$

“nothing
possible”



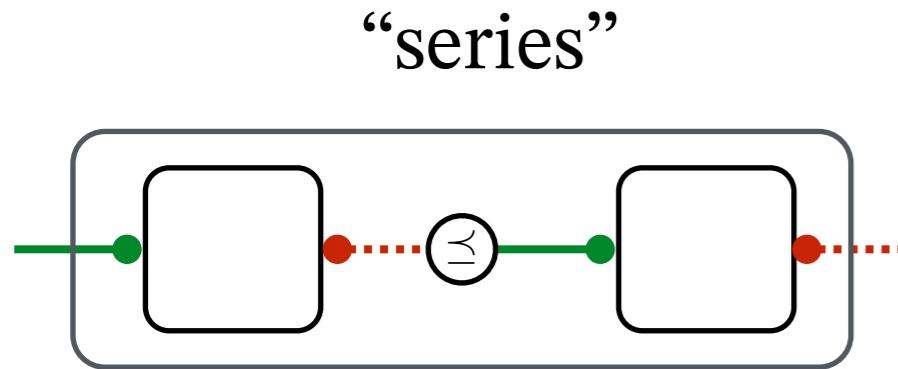
“convince two experts”



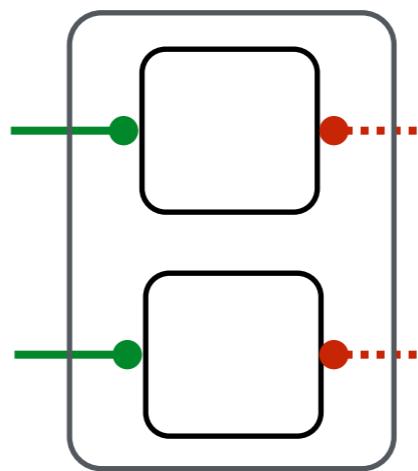
“choose any”

“everything
possible”

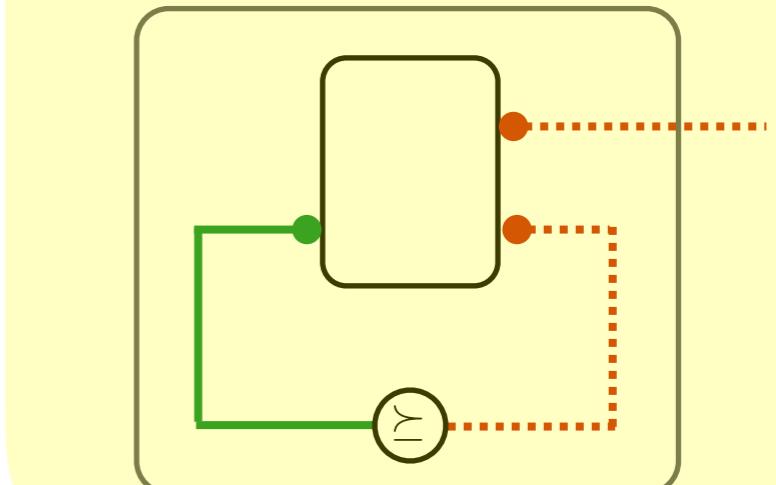
What's missing?



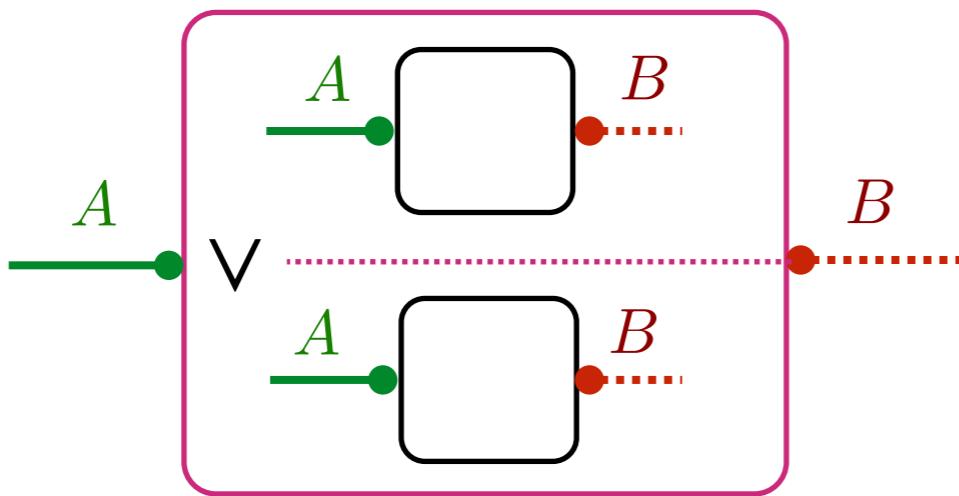
“parallel”



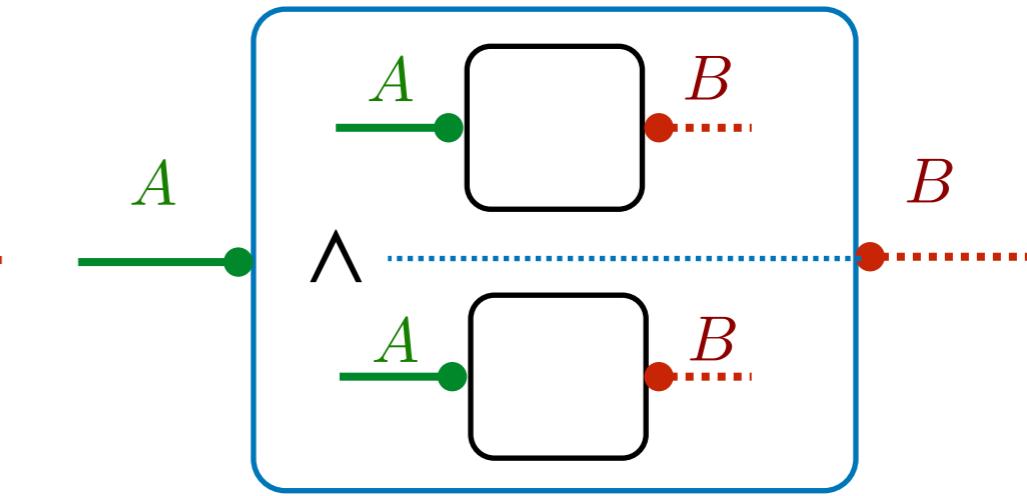
“feedback”



“choose between
two options”



“convince two experts”



Logistics, announcements

Next Week plan:

- ▶ **Saturday, January 30th at 09:00 UTC:** Office hour;
- ▶ **Monday, February 1st at 18:00 UTC:** Session 11 (Feedback);
- ▶ **Wednesday, February 3rd at 14:00 UTC:** Session 12 (Optimization);
- ▶ **Thursday, February 4th at 20:00 UTC:** Guest Lecture 4 (Prof. Michael Johnson);
- ▶ **Friday, February 5th at 1800 UTC:** Session 13 (Summary).

Question at Paolo's talk

All told, a monad in \mathbf{X} is just a monoid in the category of endofunctors of \mathbf{X}

- ▶ Given a category \mathbf{C} , the **endofunctor** category of \mathbf{C} has:
 - Objects: endofunctors $F: \mathbf{C} \rightarrow \mathbf{C}$;
 - Morphisms: natural transformations between functors.

Now, a monoid is:

- A set M ;
- An operation $(\cdot): M \times M \rightarrow M$;
- A *neutral element* $e \in M$. Can be written $e: 1 \rightarrow M$.

A monad is:

- An endofunctor $T: \mathbf{X} \rightarrow \mathbf{X}$;
- A natural transformation $\mu: T \times T \rightarrow T$;
- A natural transformation $\eta: I \rightarrow T$.