Session 11 - Feedback: Duals and Trace

Applied Compositional Thinking for Engineers

Plan

- Duals in symmetric monoidal categories
  - Tensor product and duals in linear algebra
  - String diagrams
  - Dual objects in a SMC
  - Other examples
- Traces and feedback
  - Trace in linear algebra
  - Trace in a SMC
  - Trace, generalized
  - Trace, axiomatized

Tensor and duals in linear algebra

We'll look in detail at $\mathrm{C} = \mathrm{FinVect}_{\mathbb{R}}$.

One possible symmetric monoidal structure:

- monoidal product $\otimes$ = tensor product
- monoidal unit I = the vector space $\mathbb{R}$

Recall...

- $$\begin{aligned} \otimes : \mathrm{FinVect}_{\mathbb{R}} \times \mathrm{FinVect}_{\mathbb{R}} &\longrightarrow \mathrm{FinVect}_{\mathbb{R}} \\ (V, W) &\longmapsto V \otimes W \\ (f, g) &\longmapsto f \otimes g \end{aligned}$$

- $\mathrm{I} \in \mathrm{Ob}_{\mathrm{FinVect}_{\mathbb{R}}}$ is $\mathrm{I} = \mathbb{R}$

...plus coherence data (and coherence conditions):

- $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$
- $I \otimes V \simeq V \simeq V \otimes I$
- $V \otimes W \simeq W \otimes V$

Strictification theorem: "we can think about symmetric monoidal categories as if they were strict"

- **Example**: $V = \mathbb{R}^2$, $W = \mathbb{R}^3$. $\qquad V \otimes W = ?$

  - $\dim(V \otimes W) = \dim V \cdot \dim W = 6$. So $V \otimes W \simeq \mathbb{R}^6$.
  - If $\{a_1, a_2\}$ basis of V and $\{b_1, b_2, b_3\}$ basis of W, we can take

    $$\{a_1 \otimes b_1, a_2 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_2, a_1 \otimes b_3, a_2 \otimes b_3\}$$

    as basis of $\mathbb{R}^2 \otimes \mathbb{R}^3$.
  - $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_1 \otimes b_2 \text{ "} = \text{" } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Generally and formally: $V \otimes W$ can be defined via

Generators:

$$v \otimes w \qquad\qquad v \in V,\ w \in W$$

(Take formal linear combinations: $\lambda_1 v_1 \otimes w_1 + \ \dots\ + \lambda_n v_n \otimes w_n \quad \lambda_i \in \mathbb{R}$)

Relations:

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

$$(v + v') \otimes w = v \otimes w + v' \otimes w$$

$$v \otimes (w + w') = v \otimes w + v \otimes w'$$

There are different ways to formally define the tensor product $V \otimes W$, but...

Universal property

$$\begin{tikzcd}
V \times W \arrow[d, "\otimes"'] \arrow[dr, "B"] & \\
V \otimes W \arrow[r, dashed, "\exists\,!\, f_B"'] & U
\end{tikzcd}$$

$$\{\text{Bilinear maps } V \times W \longrightarrow U\} \simeq \{\text{Linear maps } V \otimes W \longrightarrow U\}$$

$$f_B(v \otimes w) = B(v, w)$$

$$f_B(\lambda v_1 \otimes w + v_2 \otimes w_2) = \lambda f_B(v_1 \otimes w_1) + f_B(v_2 \otimes w_2)$$

Duals in $\mathrm{FinVect}_{\mathbb{R}}$: $V^* = \mathrm{Hom}(V, \mathbb{R})$.

Given basis $e_1, ..., e_n$ of V, the **dual basis** $e_1^*, ..., e_n^*$ of V^* is characterized by

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Example:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3, \quad e_2^* = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \in (\mathbb{R}^3)^*$$

$$e_2^*(e_1) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0, \quad e_2^*(e_2) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$$

Fact: $V^* \otimes V \simeq \mathrm{Hom}(V,V)$ (and $\simeq V \otimes V^*$):

$$\begin{cases} V^* \otimes V & \longrightarrow & \mathrm{Hom}(V,V) \\ e_j^* \otimes e_j & \longmapsto & A_{i,j} : x \mapsto e_j^*(x)e_i \end{cases}$$

$$e_j^* \otimes e_i \longmapsto \begin{array}{cc} j^{\text{th}} \text{ column} & \\ \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} & i^{\text{th}} \text{ row} \end{array}$$

Example:

$$e_2^* = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e_2^*(e_1)e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2^*(e_2)e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad e_2^*(e_3)e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A_{3,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\mathrm{Hom}(V,V) \simeq V^* \otimes V \quad (\text{and } \simeq V \otimes V^*)$:

$$\begin{cases} \mathrm{Hom}(V,V) & \longrightarrow & V^* \otimes V \\ A & \longmapsto & \sum_{i,j} a_{i,j}\, e_j^* \otimes e_i \end{cases} \qquad \text{where } a_{i,j} = e_i^*(Ae_j)$$

$\mathrm{Hom}(V, V) \simeq V^* \otimes V \quad$ (and $\simeq V \otimes V^*$):

$$\left\{\begin{array}{ccc} \mathrm{Hom}(V,V) & \longrightarrow & V^* \otimes V \\ A & \longmapsto & \sum_{i,j} a_{i,j}\, e_j^* \otimes e_i \end{array}\right. \qquad \text{where } a_{i,j} = e_i^*(Ae_j)$$

Trace: $Tr : \mathrm{Hom}(V, V) \longrightarrow \mathbb{R}, \quad A \longmapsto Tr(A)$

Evaluation: $\epsilon_V : V^* \otimes V \longrightarrow \mathbb{R}, \quad l \otimes v \longmapsto l(v)$

$$\left\{\begin{array}{ccccc} \mathrm{Hom}(V,V) & \longrightarrow & V^* \otimes V & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \sum_{i,j} a_{i,j}\, e_j^* \otimes e_i & \longmapsto & \sum_{i,j} a_{i,j}\, e_j^*(e_i) = \sum_i a_{i,i} \end{array}\right.$$

Evaluation: $\epsilon : V^* \otimes V \longrightarrow \mathbb{R}$

Coevaluation: $\eta_V : \mathbb{R} \longrightarrow V \otimes V^*$

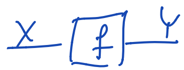
$$\eta_V : \begin{cases} \mathbb{R} & \longrightarrow & V \otimes V^* \\ \lambda & \longmapsto & \lambda \sum_i e_i \otimes e_i^* \end{cases}$$

$$\begin{cases} \mathbb{R} & \longrightarrow & V \otimes V^* \simeq \mathrm{Hom}(V, V) \\ \lambda & \longmapsto & \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} \end{cases}$$

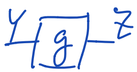
String diagrams

Series composition:

Given $f : X \to Y$

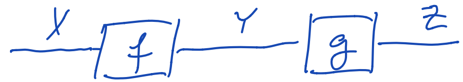


and $g : Y \to Z$



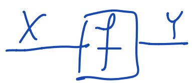
$\rightsquigarrow$

$f ; g : X \to Z$

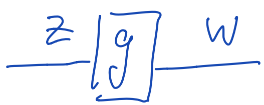


Parallel composition:

Given $f : X \to Y$



and $g : Z \to W$

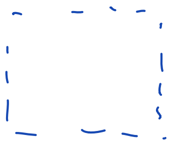


$\rightsquigarrow$

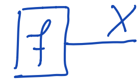
$f \otimes g : X \otimes Y \to Z \otimes W$

Unit object:

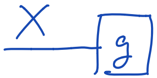
$I \in \mathrm{Ob}_{\mathcal{C}}$



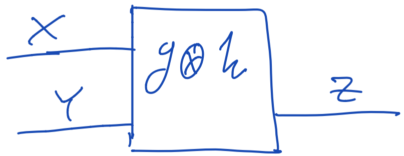
$f : I \to X$



$g : X \to I$

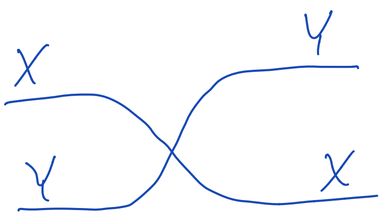


$h : Y \to Z$



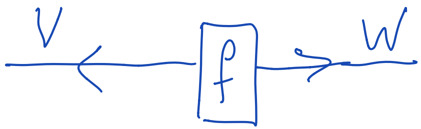
Symmetry:

$\sigma_{X,Y} : X \otimes Y \simeq Y \otimes X$

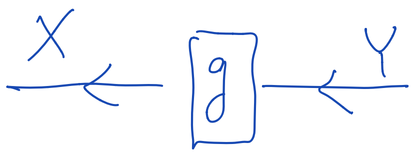


New: we'll add a "**flow direction**" to our strings/wires, in addition to the directionality of reading from left to right and top to bottom.

Meaning:



means $f : V^* \longrightarrow W$

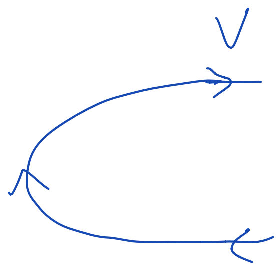


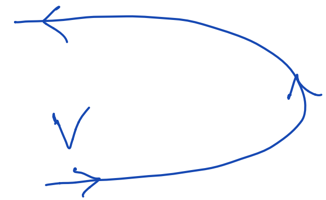
means $g : X^* \longrightarrow Y^*$

Need direction of flow to match for composition:

not composable!

New: we add two new symbols, for **coevalution** and **evaluation**.

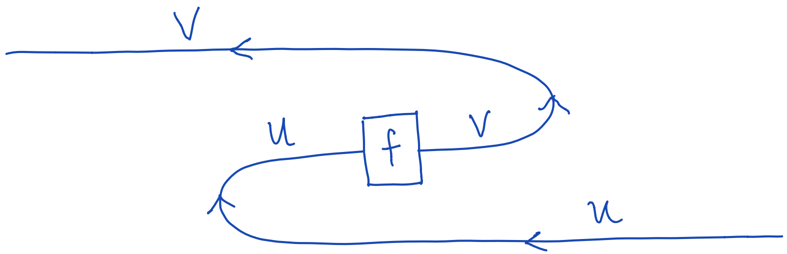




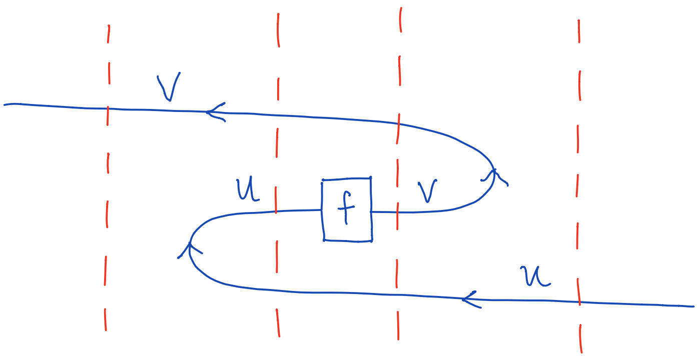
$$\mathbb{R} \xrightarrow{\eta_V} V \otimes V^* \qquad\qquad V^* \otimes V \xrightarrow{\varepsilon_V} \mathbb{R}$$

Then we can do things like this:

$f : U \to V$



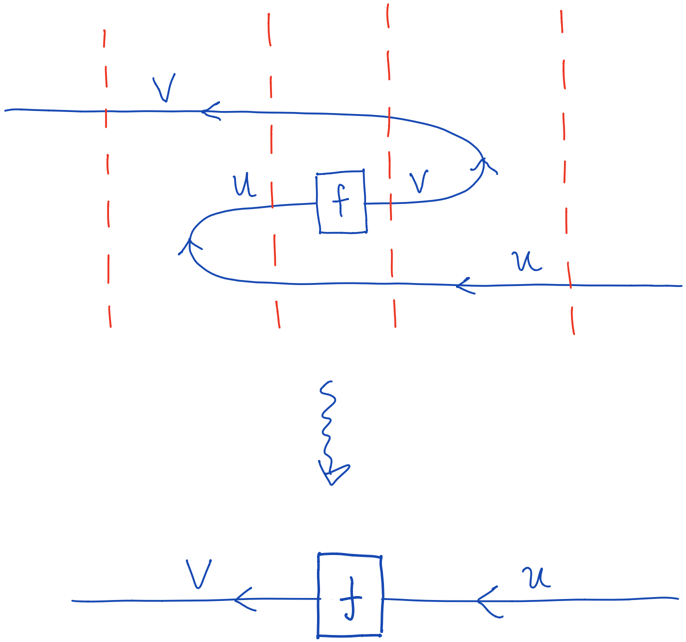
How to read this?



$$V^* \cong \begin{matrix} V^* \\ \otimes \\ I \end{matrix} \xrightarrow{\begin{matrix} id \\ \eta_U \end{matrix}} \begin{matrix} V^* \\ \otimes \\ U \\ \otimes \\ U^* \end{matrix} \xrightarrow{\begin{matrix} id \\ f \\ id \end{matrix}} \begin{matrix} V^* \\ \otimes \\ V \\ \otimes \\ U^* \end{matrix} \xrightarrow{\begin{matrix} \varepsilon_V \\ id \end{matrix}} \begin{matrix} I \\ \otimes \\ U^* \end{matrix} \cong U^*$$

$$id_{V^*} \otimes \eta_U \; ; \; id_{V^*} \otimes f \otimes id_{U^*} \; ; \; \varepsilon_V \otimes id_{U^*}$$

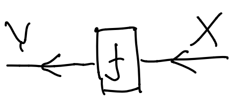
Exercise: show that this composition is precisely $f^* : U^* \to V^*$.



$$id_{V^*} \otimes \eta_U \,;\, id_{V^*} \otimes f \otimes id_{U^*} \,;\, \varepsilon_V \otimes id_{U^*}$$

$$=$$

$$f^*$$

Definition: 

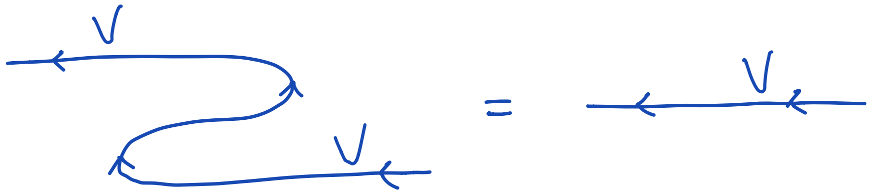
$:= \quad f^* : Y^* \longrightarrow X^*$

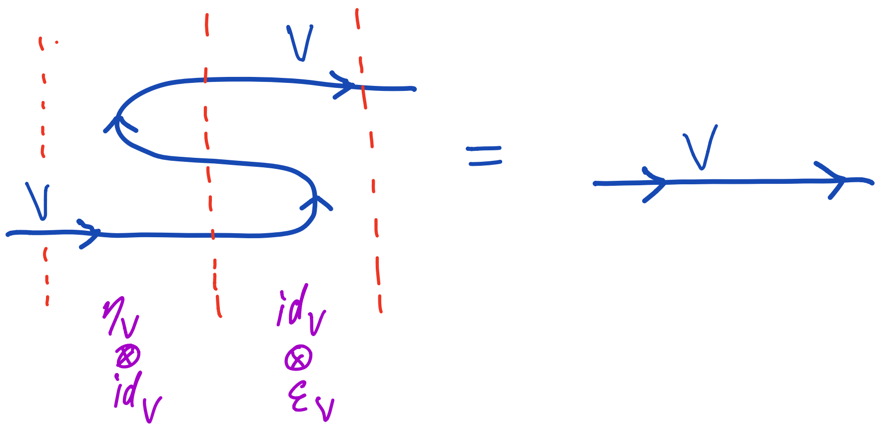
We can do calculations and proofs via deforming string diagrams, following certain rules.

Is this black magic ?
How to do we know that everything is rigorous?

Joyal and Street, in the 90’s, put string diagrams on a rigorous footing…

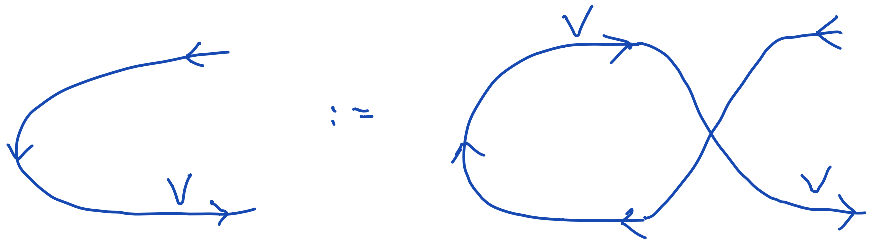
Example of some basic rules/moves: Zig-zag identities (aka “snake equations”)



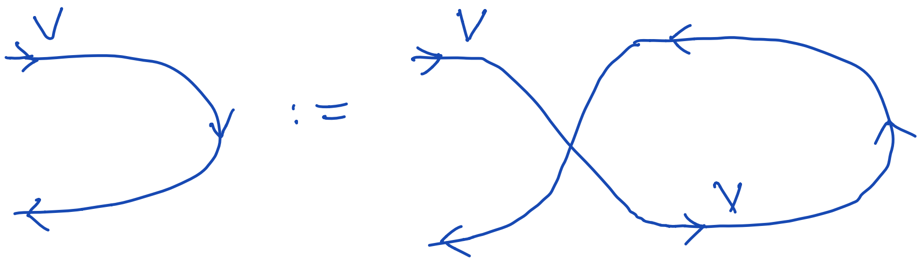


$$\eta_V \otimes \mathrm{id}_V \,;\, \mathrm{id}_V \otimes \epsilon_V = \mathrm{id}_V$$

Using η_V, ϵ_V, and the symmetry operation, we also get:



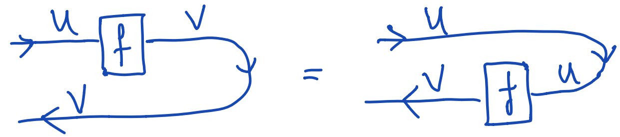
$$\eta_{V^*} := \eta_V \,;\, \sigma_{V,V^*}$$



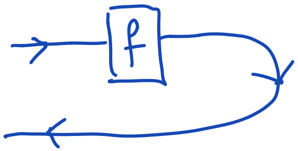
$$\epsilon_{V^*} := \sigma_{V,V^*} \,;\, \epsilon_V$$

Example proof using string diagrams:

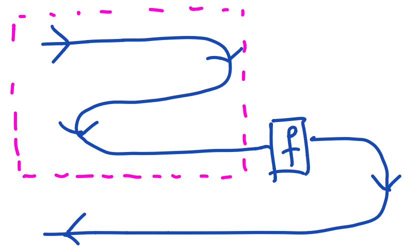
Lemma: given $f : U \to V$



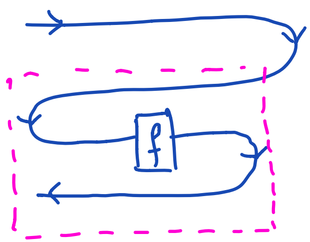
Proof:



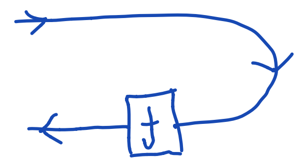
zig zag =



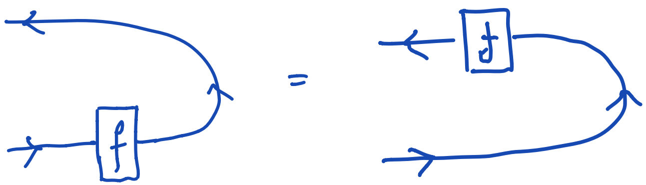
stretching =

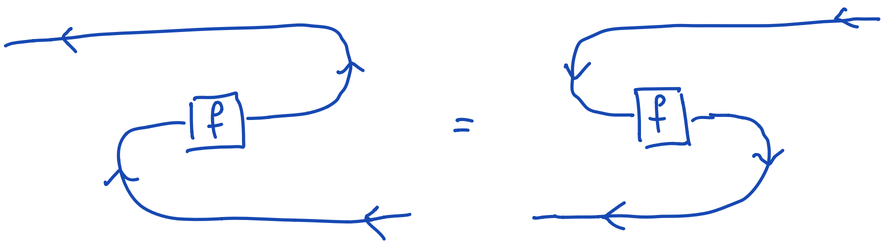


=



In a similar way, one can prove for example:





(both represent f^*)

etc.

Dual objects in a SMC

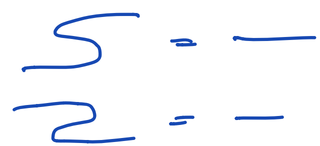
Definition: Let C be a (strict) symmetric monoidal category, and $X \in \mathrm{Ob}_{\mathrm{C}}$. A ***right dual*** of X is defined by the following data, satisfying the following conditions.

Data:

- An object X^* of C
- A morphism $\eta_X : I \to X \otimes X^*$, called coevaluation 
- A morphism $\epsilon_X : X^* \otimes X \to I$, called evaluation 

Conditions:

- $\eta_X \otimes \mathrm{id}_X \fatsemi \mathrm{id}_X \otimes \epsilon_X = \mathrm{id}_X$
- $\mathrm{id}_{X^*} \otimes \epsilon_X \fatsemi \eta_X \otimes \mathrm{id}_{X^*} = \mathrm{id}_X$



In a symmetric monoidal category:

- Left duals are defined analogously.
- If an object X admits a right dual, then this can be made to be a left dual, too. So we can just speak of duals.
- If an object admits right/left duals, it is called ***dualizable***.
- A symmetric monoidal category is called ***compact closed*** if every object is dualizable.

In a symmetric monoidal category:

Proposition: Given an object X, if Y and Y' are both (right) duals to X, then $Y \simeq Y'$.

Example: $\mathbf{C} = \mathbf{FinVect}_{\mathbb{R}}$. Both V and V^{**} are (right) duals to V^*.

Proposition: Consider the SMC $\mathbf{C} = \mathbf{Vect}_{\mathbb{R}}$. An object V of $\mathbf{C}$ is dualizable if and only if V is finite-dimensional.

Remark: "dual" in the SMC sense $\neq$ "dual" in linear algebra sense

Intuition: "dualizable" = "finite" in some sense

Other examples

Example: $\langle \mathbf{Rel}, \times, \{*\}, \sigma \rangle$ is compact closed.

Given a set X, its dual is $X^* = X$.

$$\eta_X : \{*\} \longrightarrow X \times X, \quad \eta_X = \{\langle *, \langle x, x' \rangle \rangle \in \{*\} \times (X \times X) \mid x = x'\}$$

$$\epsilon_X : X \times X \longrightarrow \{*\}, \quad \epsilon_X = \{\langle \langle x, x' \rangle, * \rangle \in (X \times X) \times \{*\} \mid x = x'\}$$

Example: Boolean profunctors form a compact closed category, with $\otimes = \times$ and $I = \{*\}$.

Given a poset $X = \langle X, \leq \rangle$, its dual is the opposite poset $X^{op} = (\langle X, \leq \rangle)^{op}$.

$\eta_X : \{*\} \to X \times X^{op}$:

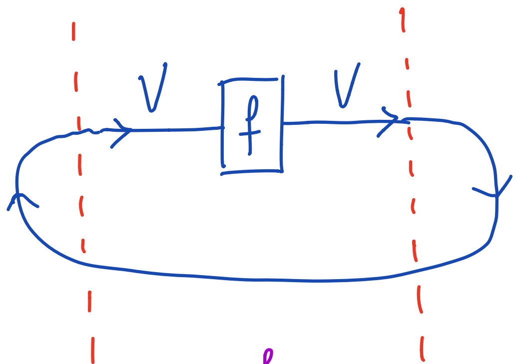
$$\{*\}^{op} \times (X \times X^{op}) \longrightarrow \text{Bool}, \ \langle *, x, x' \rangle = \top \text{ iff } x \geq_X x'$$

$\epsilon_X : X^{op} \times X \longrightarrow \{*\}$:

$$(X^{op} \times X)^{op} \times \{*\} \longrightarrow \text{Bool}, \ \langle x, x', * \rangle = \top \text{ iff } x \geq_X x'$$

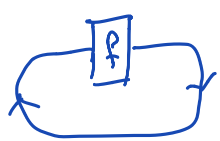
Trace in linear algebra

Given $f : V \to V$, how can we compute its trace in a CT way?



$$\mathbb{R} \xrightarrow{\eta_V} \begin{matrix}V\\ \otimes \\ V^*\end{matrix} \xrightarrow[\mathrm{id}]{f} \begin{matrix}V\\ \otimes \\ V^*\end{matrix} \xrightarrow{\varepsilon_V} \mathbb{R}$$

$$\lambda \longmapsto \lambda \sum_i e_i \otimes e_i^* \longmapsto \lambda \sum_i f e_i \otimes e_i^* \longmapsto \lambda \sum_i e_i^*(f e_i) = \lambda \mathrm{Tr}(f)$$

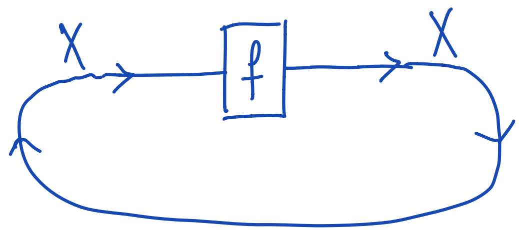


$$\left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ \lambda \longmapsto \lambda \mathrm{Tr}(f) \end{array} \right.$$

Trace in a SMC

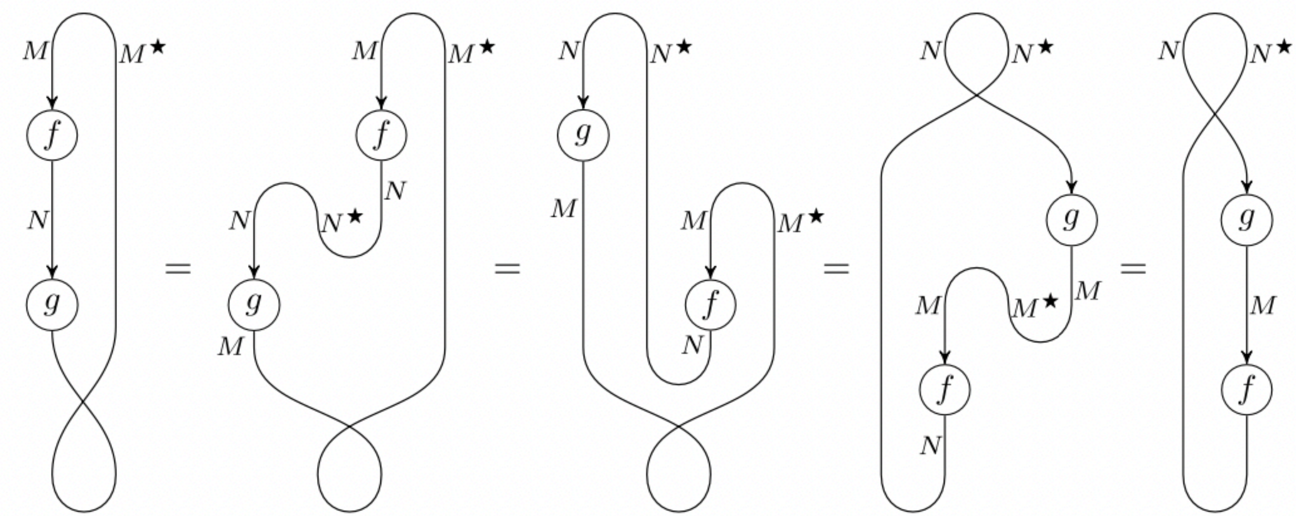
Let **C** be a SMC, X a dualizable object, and $f : X \to X$ an endomorphism. The **trace** of f is the composite

$$I \xrightarrow{\eta_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X \otimes X^* \xrightarrow{\sigma} X^* \otimes X \xrightarrow{\epsilon_X} I$$



$Tr(f) \in Hom_{\mathcal{C}}(I, I)$

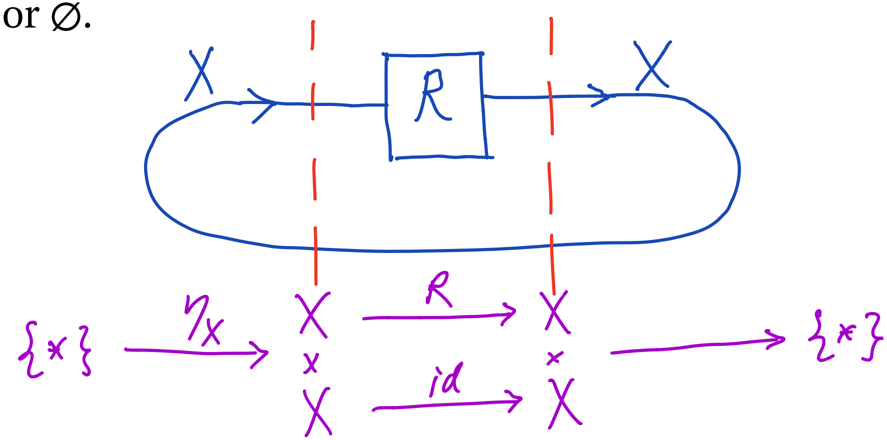
Proposition: Let M, N be dualizable objects of a SMC $\mathbf{C}$. If $f : M \to N$ and $g : N \to M$, then $\mathrm{Tr}(f \mathbin{⨾} g) = \mathrm{Tr}(g \mathbin{⨾} f)$.



K. Ponto, M. Shulman, Traces in monoidal categories, *Expo. Math.* 32 (2014)

Example: Let $\mathbf{C} = \mathbf{Rel}$, with cartesian product $\times$ as monoidal product and $I = \{*\}$. The dual of a set X is the set X again.

Let $R : X \to X$ be a relation. The trace $\mathrm{Tr}(R)$ is a relation $\{*\} \to \{*\}$, so it is either $\{(*, *)\}$ or $\varnothing$.



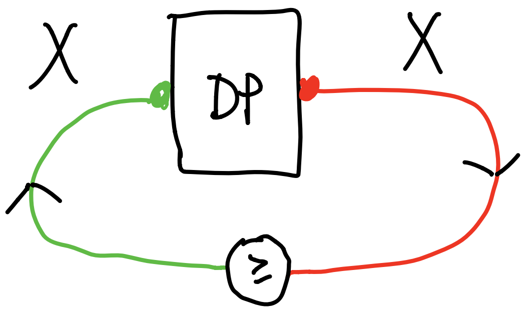
$$\mathrm{Tr}(R) = \begin{cases} (*, *) & \text{if } \exists\, (x, y) \in R : x = y \\ \varnothing & \text{else} \end{cases}$$

Example: Let $\mathbf{C} = \mathbf{DP}$. Cartesian product $\times$ is monoidal product, $I = \{*\}$. The dual X^* of a poset X is its opposite poset.

Let $\phi : X \to X$ be a boolean profuntor. Recall that $\phi^{-1}(\top)$ is the "feasible subset" of $X \times X$.

The trace $\mathrm{Tr}(\phi)$ is a boolean profunctor $\{*\} \to \{*\}$, so it corresponds to either $\{(*, *)\}$ or $\varnothing$.

$$\mathrm{Tr}(\phi) = \begin{cases} (*, *) & \text{if } \exists \langle x, y \rangle : \phi(\langle x, y \rangle) = \top \wedge x \geq y \\ \varnothing & \text{else} \end{cases}$$



Trace, generalized

Definition: Let U be a dualizable object in a SMC $\mathbf{C}$. Given a morphism in $\mathbf{C}$

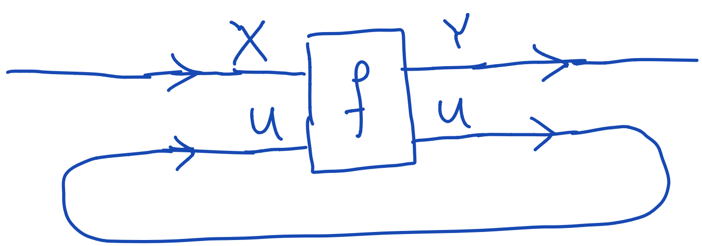
$$f : X \otimes U \longrightarrow Y \otimes U$$

its **trace** over U is the morphism

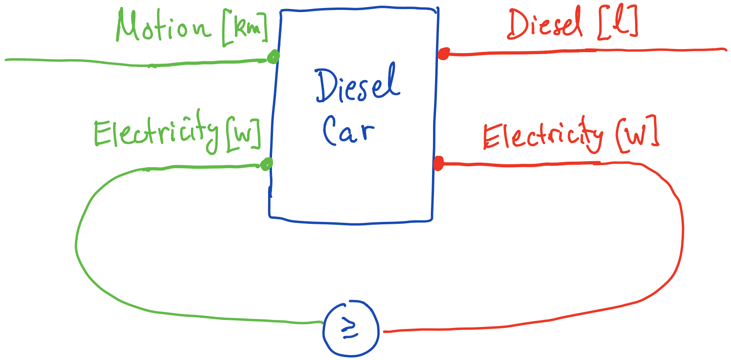
$$\mathrm{Tr}^{U}_{X,Y}(f) : X \longrightarrow Y$$

given by the composite

$$X \xrightarrow{\mathrm{id}_X \otimes \eta_U} X \otimes U \otimes U^* \xrightarrow{f \otimes \mathrm{id}_{U^*}} Y \otimes U \otimes U^* \xrightarrow{\mathrm{id}_Y \otimes \sigma_{U,U^*}} Y \otimes U^* \otimes U \xrightarrow{\mathrm{id}_Y \otimes \epsilon_U} Y$$



Example:



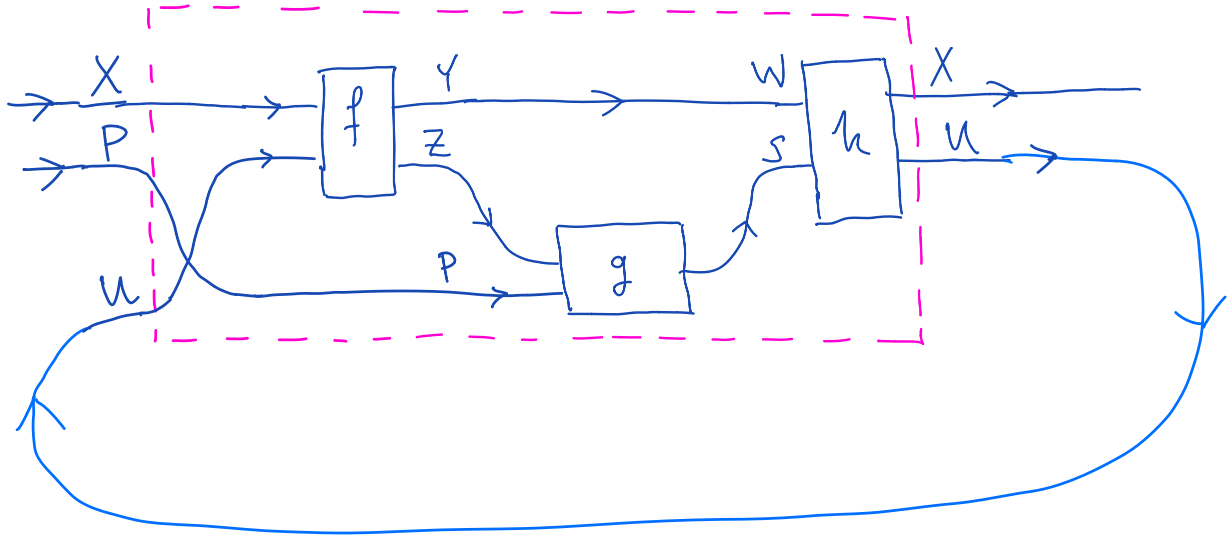
X = Motion
Y = Diesel
U = Electricity

$\phi : X \times U \longrightarrow Y \times U$

Boolean profunctor
"Diesel Car"

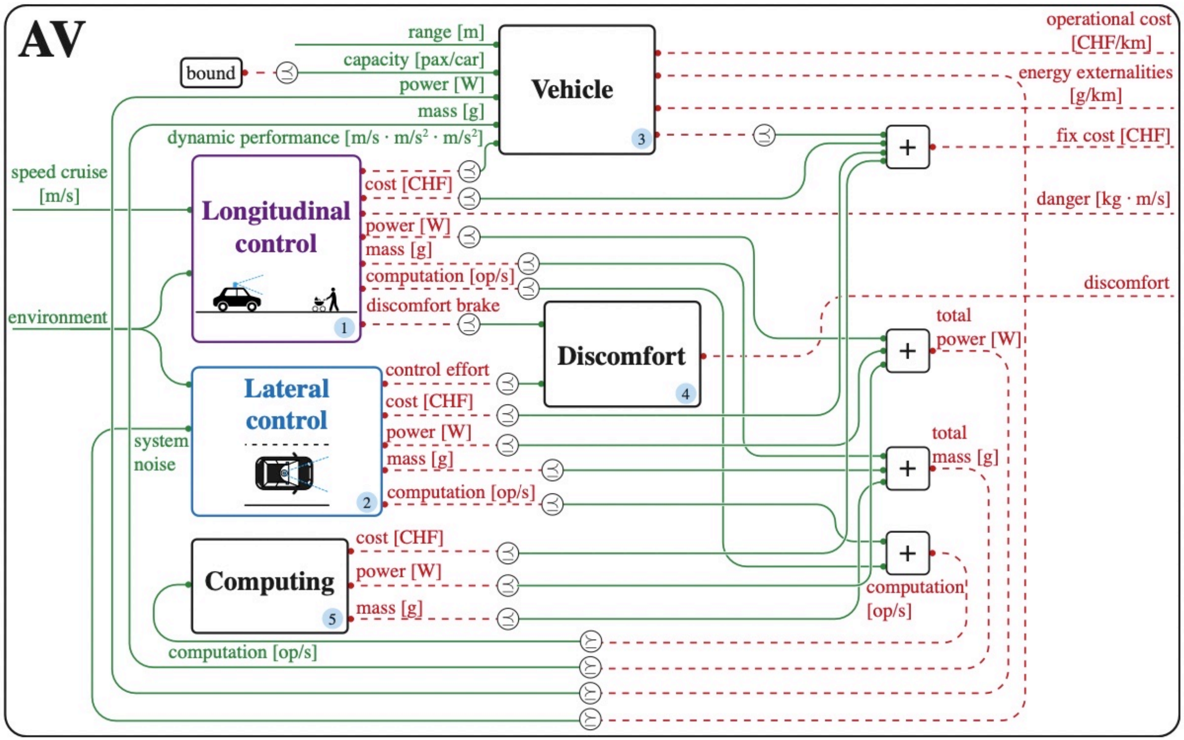
$$\phi : X \times U \longrightarrow Y \times U$$

$$\mathrm{Tr}(\phi)^{-1}(\top) = \{\langle x, y\rangle \in X \times Y \mid \exists \langle u, u'\rangle : \phi(\langle\langle x, u\rangle, \langle y, u'\rangle\rangle) = \top \ \wedge \ u \geq u'\}$$



$$\mathrm{id}_X \otimes \sigma_{u,p} \,;\, f \otimes \mathrm{id}_p \,;\, \mathrm{id}_Y \otimes g \,;\, h$$

Now we can interpret such diagrams:



Trace, axiomatized

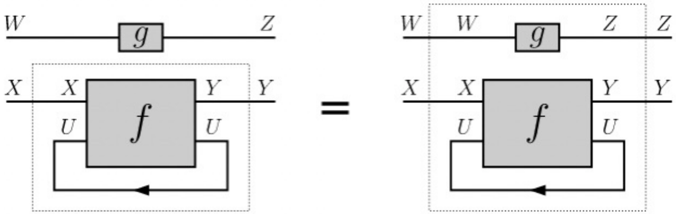
Definition: A **traced symmetric monoidal category** is a SMC $\mathbf{C}$ together with a family of functions

$$\mathrm{Tr}^{U}_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X \otimes U, Y \otimes U) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(X, Y), \qquad f \longmapsto \mathrm{Tr}^{U}_{X,Y}(f)$$

satisfying a number of conditions (which we omit here).

We also say: $\mathbf{C}$ is equipped with a **trace structure**.

$$g \otimes \mathrm{Tr}^{U}_{X,Y}(f) = \mathrm{Tr}^{U}_{W\otimes X, Z\otimes Y}(g \otimes f)$$



C.f. Wikipedia page for "traced monoidal category" to see the conditions + diagrams illustrating them.

Example: $(\mathbf{Rel}, +, \varnothing)$ is symmetric monoidal but *not compact closed*. There is however the following trace structure. Given a relation $R : X + U \longrightarrow Y + U$,

$$\mathrm{Tr}^{U}_{X,Y}(R) : X \longrightarrow Y$$

is the relation

$$\{\langle x, y\rangle \in X \times Y \mid \exists\, n \geq 0,\ \exists\, u_1, \ldots, u_n \in U \ :\ xRu_1Ru_2 \ldots u_nRy\}$$