Applied Compositional Thinking for Engineers

Spring 2021

Recital 1

Today's plan

- Examples of:
  - Semigroups
  - Monoids
  - Groups

- Today at **16:15 CET** at http://bit.ly/3rbg6rW we have an **office hour**

- Google doc for today available at: http://bit.ly/3eh2Oqv

Exercises

- We are going to have exercises on a weekly basis:
  - Theoretical exercises (due in 10 days from publication)
  - Programming exercises in Python, autograded with GitHub Classroom

    Due on the last day of the course
- We will announce them soon

Magmas

Definition (Magma). A *magma* **S** is a set **S**, together with a binary operation

$$\fatsemi : \mathbf{S} \times \mathbf{S} \to \mathbf{S},$$

- **Example (cross-product):**

$$S = \mathbb{R}^3$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \fatsemi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ -(az - bx) \\ ay - bx \end{pmatrix}$$

Semigroups

Definition (Semigroup). A *semigroup* **S** is a set **S**, together with a binary operation

$$⨾ : \mathbf{S} \times \mathbf{S} \to \mathbf{S},$$

called *composition*, which satisfies the *associative* law:

$$(x ⨾ y) ⨾ z = x ⨾ (y ⨾ z)$$

for all $x, y, z \in \mathbf{S}$.

- **Warmup example:** let's look at **max** on natural/real numbers:

$$x ⨾ y \doteq \max(x, y)$$

- What do we need to check?

Semigroups

Definition (Semigroup). A *semigroup* **S** is a set $\mathbf{S}$, together with a binary operation

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- **Warmup example:** let's look at **max** on natural/real numbers:

$$x ⨾ y \doteq \max(x, y)$$

- What do we need to check? Just **associativity**:

$$\max(\max(x, y), z) = \max(x, \max(y, z))$$

Semigroups

- **Example (strings):**
  - $A = \{\circ, \bullet\}$
  - S is the set of all non-empty strings of A

    $\circ\circ\bullet\circ\bullet$

  - $\left.\begin{array}{c} \circ\circ\bullet\circ \quad ; \quad \circ\bullet\bullet\bullet \\ \circ\circ\bullet\circ\bullet\bullet\bullet \end{array}\right\}$ associative

- **Example (true-false)**

  $S = \{\text{false}, \text{true}\}$, $\langle S, \overset{\text{"AND"}}{\wedge} \rangle$

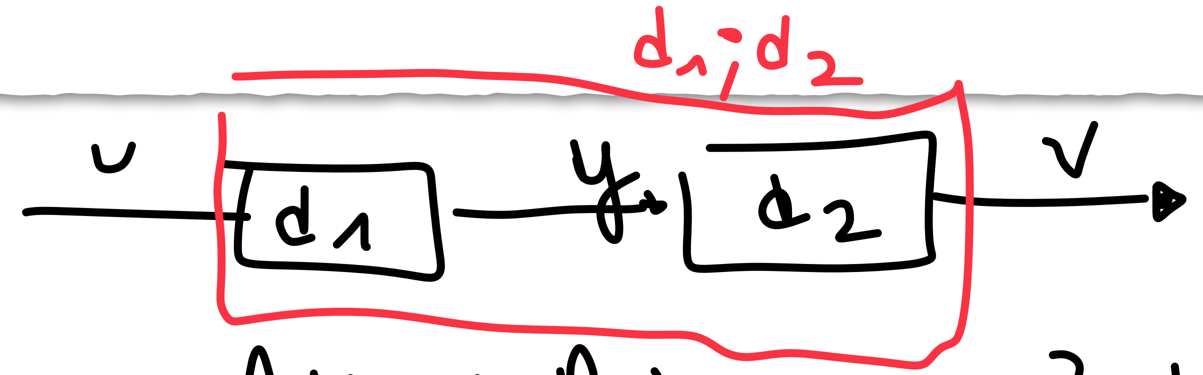
$\wedge$	false	true
false	false	false
true	false	true

$\Rightarrow \quad \perp \quad \top$

Semigroups

- **Example (discrete-time linear systems)**

> **Definition** (Discrete-time linear systems). A discrete-time linear time-invariant proper open system is defined by three matrices A, B, C. Together they give a recurrence of the type
>
> $$\begin{aligned} x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k \\ y_k &= \mathbf{C}x_k \end{aligned}$$
>
> If x has dimension $n \geq 1$, u dimension $m \geq 1$ and y dimension $p \geq 1$, then A has dimension $n \times n$, B has dimension $n \times m$, C has dimension $p \times n$.



- **Composition:**

d_1 :
$$x_{k+1} = Ax_k + Bu_k \qquad z_{k+1} = Ez_k + Fy_k$$
$$y_k = Cx_k \qquad v_k = Gz_k$$

$d_1 ; d_2$:
$$\begin{pmatrix} x_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ FC & E \end{pmatrix} \begin{pmatrix} x_k \\ z_k \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u_k$$

$$v_k = G \cdot z_k$$

Semigroups

- **Associativity:**

$d_3 : \quad s_{k+1} = H s_k + I v_k$

$w_k = J s_k$

$$(d_1 ; d_2) ; d_3 : \begin{bmatrix} x_{k+1} \\ z_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{pmatrix} A & 0 & 0 \\ FC & E & 0 \\ 0 & IG & H \end{pmatrix} \begin{bmatrix} x_k \\ z_k \\ s_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} v_k$$

$$w_k = J s_k$$

$$\|$$

$$d_1 ; (d_2 ; d_3) \quad \Rightarrow \text{Exercise}$$

Semigroups

- **Example (growing plants)**

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

let $T : X \longrightarrow X$ be "development"

$$\left.\begin{aligned} T(\text{sprout}) &= \text{young} \\ T(\text{young}) &= \text{mature} \end{aligned}\right\} \text{3 months}$$

$$\vdots$$

$$T(\text{old}) = \text{dead}$$

$$T(\text{dead}) = \text{dead}$$

$T \circ T \circ T : X \longrightarrow X \implies$ nine months

$\langle S = \{T^n \mid n \in \mathbb{N}\}, \circ \rangle$

Monoids

- A **monoid** is a semigroup with a neutral element.

> **Definition** (Monoid). A *monoid* **M** is:
>
> Constituents
>
> 1. a set $\mathbf{M}$;
> 2. a binary operation $⨾ : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$;
> 3. a specified element $\mathrm{id} \in \mathbf{M}$, called *neutral element*.
>
> Conditions
>
> 1. Associative law: $(x ⨾ y) ⨾ z = x ⨾ (y ⨾ z)$;
> 2. Neutrality Laws: $\mathrm{id} ⨾ x = x = x ⨾ \mathrm{id}$.

- **Warmup example:**

Monoids

- **Example (*free* monoid)**

$$\bullet\circ\bullet\bullet \,;\, \circ\circ\circ$$

$$\bullet\circ\bullet\bullet\circ\circ\circ$$

$$\bullet\circ\bullet\bullet \,;\, \langle\rangle = \bullet\circ\bullet\bullet$$

- **Example (true-false)**

$$\langle \{\text{false}, \text{true}\}, \wedge, \text{true} \rangle$$

$$\text{true} \wedge x = x$$

Monoids

- **Example (continuous dynamical systems)**

> **Definition** (Continuous-time dynamical system). A dynamical system on $\mathbb{R}^n$ may be defined by a function
>
> $$f : \mathbb{R}^n \to \mathbb{R}^n.$$
>
> A trajectory of a dynamical system is a function $x : \mathbb{R} \to \mathbb{R}^n$ such that
>
> $$\dot{x}_t = f(x_t).$$
>
> We use the notation $\dot{x}$ to abbreviate $\mathrm{d}x/\mathrm{d}t$.

- **Composition:**

· Assume that for any x_0 we have exactly one traj. starting at x_0

· Given x: where are we after δ?
$T_\delta : \mathbb{R}^n \to \mathbb{R}^n$ (transition)

$$\boxed{T_{\delta_1} \fatsemi T_{\delta_2} = T_{\delta_1+\delta_2}}$$

$$T_{\delta_1} \fatsemi (T_{\delta_2} \fatsemi T_{\delta_3}) =$$

$$T_{\delta_1} \fatsemi T_{\delta_2+\delta_3} = T_{\delta_1+\delta_2+\delta_3}$$

$$= T_{\delta_1+\delta_2} \fatsemi T_{\delta_3}$$

$$= T_{\delta_1} \fatsemi (T_{\delta_2} \fatsemi T_{\delta_3})$$

$$T_{\delta=0} \fatsemi T_\delta = T_\delta = T_\delta \fatsemi T_0$$

$$Tr = \{T_\delta \mid \delta \in \mathbb{R}_{\geq 0}\}$$

$$\langle Tr, \fatsemi, T_0 \rangle$$

$$Tr \times Tr \longrightarrow Tr$$

Monoids

- **Associativity**

- **Neutrality**

Monoids

- **Example (discrete-time linear systems):**

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

Monoids

- **Example (cross-product):**

Does the magma form a monoid with appropriate neutral el?

Groups

Definition (Group). A *group* is a monoid together with an "inverse" operation. In more detail, a group $\mathbf{M}$ is

Constituents

1. a set $\mathbf{M}$;
2. a binary operation $\fatsemi : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$, called *composition*;
3. a specified element $\mathrm{id} \in \mathbf{M}$;
4. a map $\mathrm{inv} : \mathbf{M} \to \mathbf{M}$ called "inverse".

Conditions

1. Associative law: $(x \fatsemi y) \fatsemi z = x \fatsemi (y \fatsemi z)$;
2. Neutrality Laws: $\mathrm{id} \fatsemi x = x = x \fatsemi \mathrm{id}$.
3. Inverse law:

$$\mathrm{inv}(x) \fatsemi x = \mathrm{id} = \cancel{\mathrm{inv}(x) \fatsemi x} \quad x \fatsemi \mathrm{inv}(x)$$

- **Warmup example (taking the negative):**

$$\langle \mathbb{Z}, +, 0 \rangle, \quad \mathrm{inv}(x) := -x \quad \forall x \in \mathbb{Z}$$

- **Warmup example (taking the inverse):**

$$\langle \mathbb{R} \setminus \{0\}, \cdot, 1 \rangle, \quad \mathrm{inv}(x) := \frac{1}{x} \quad \forall x \in \mathbb{R}$$

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2. Neutrality Laws: $\mathrm{id} \fatsemi x = x = x \fatsemi \mathrm{id}$.
3. Inverse law:

$$\mathrm{inv}(x) \fatsemi x = \mathrm{id} = \mathrm{inv}(x) \fatsemi x$$

- **Example (true-false):**

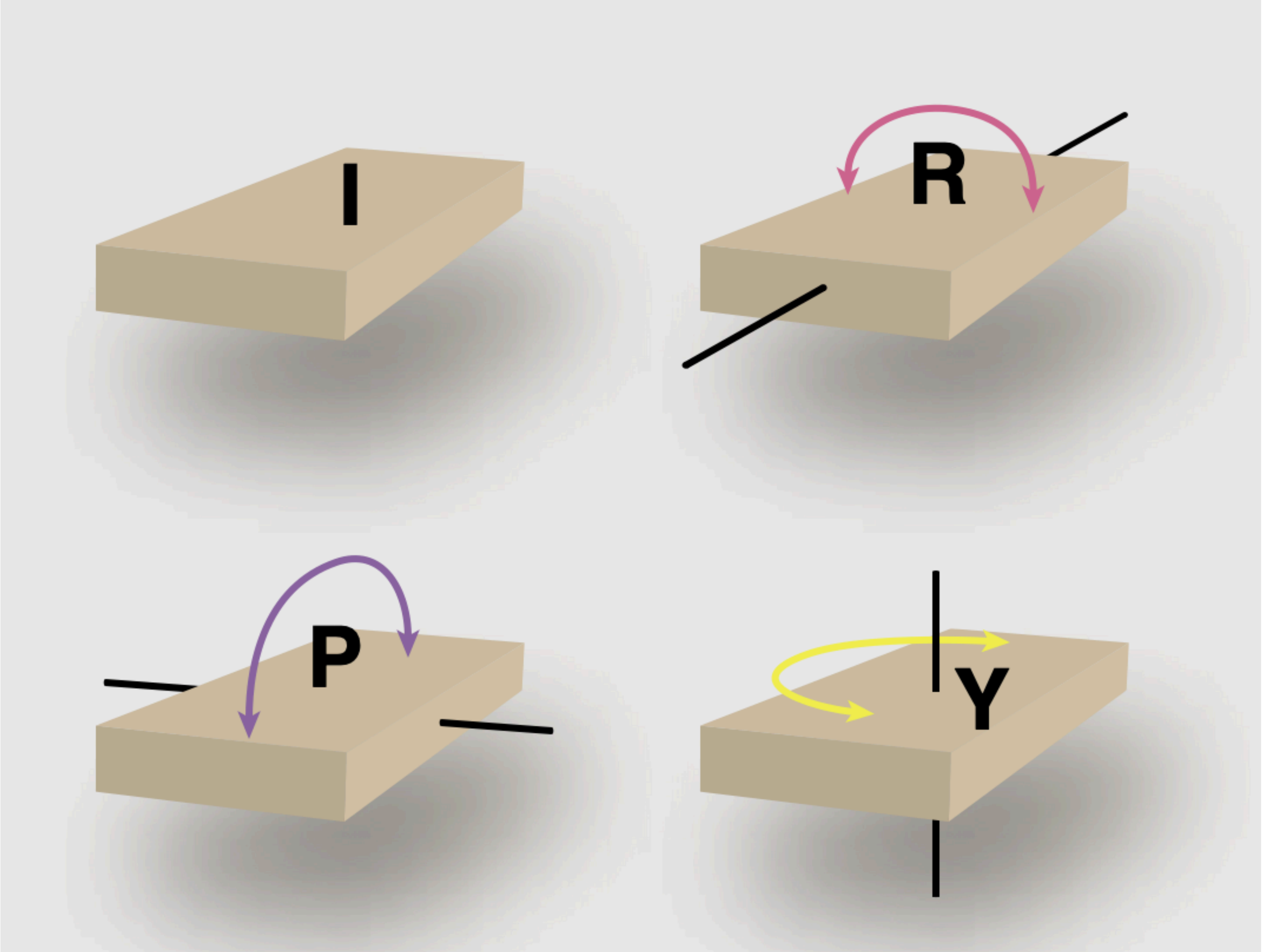
$$\langle \{\mathrm{false}, \mathrm{true}\}, \wedge, \mathrm{true} \rangle$$

$$\mathrm{false} \wedge \mathrm{inv}(\mathrm{false}) = \mathrm{true}$$

$$\lightning$$

Groups

- **Example (group theory in the bedroom)**
- Consider four configurations of your mattress:



- This is called the **Klein Vierergroup (4-group):**

• each op. is its inverse

• neutral is I

Groups

- **Example (orthogonal matrices)**

see book

-

Groups

- **Example (free group):**

See book

Groups

- **Example (properties):**

See exercises

Summary

- One way to define algebraic structure is by **refinement**:
  progressively add more properties that the structure must satisfy.

- Today we defined these structures:
  - A **magma** is a set with a **binary operation**.
  - A **semigroup** is a set with an **associative** operation
  - A **monoid** is a semigroup **with a neutral element**.
  - A **group** has an “**inverse” operation.**

- The other direction, which require more imagination creativity, is **generalization,** in the sense of **imagining a broader structure that contains the current structure as a particular case.**