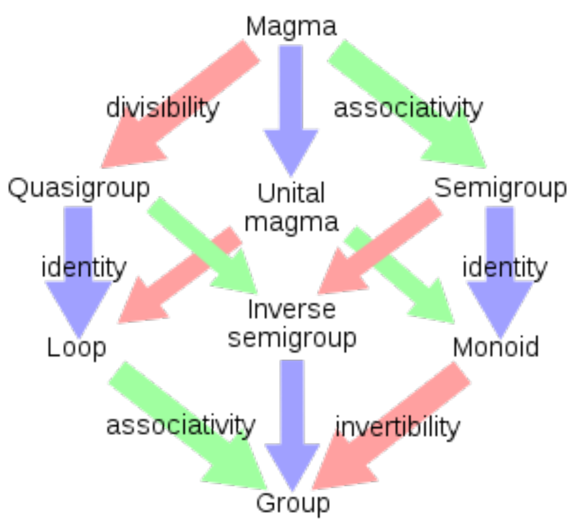
Applied Compositional Thinking for Engineers

Spring 2021

Semicategories, Categories

Building in layers

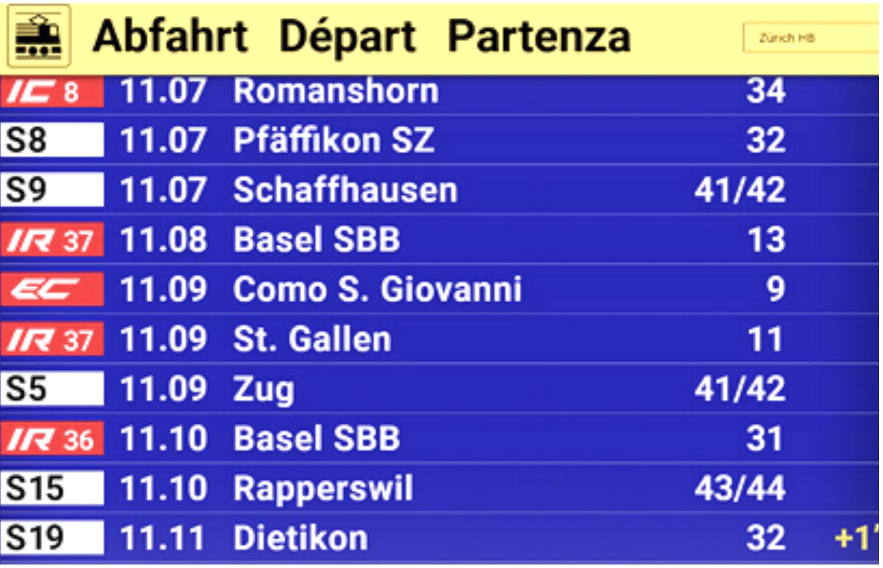
- Last week:
  - A **magma** is a set with a **binary operation** of *composition*.
  - A **semigroup** is a magma whose composition is **associative.**
  - A **monoid** is a semigroup **with a neutral element**.
  - A **group** has is a monoid with an "**inverse" operation.**



Building in layers

- Today: we will *generalize*, rather than *refine*.

$$
\begin{tikzcd}
 & \textit{Categories} & \\
\textit{Monoids} \arrow[ur, "?"] & & \textit{Semicategories} \arrow[ul, "\textit{add identities}"'] \\
 & \textit{Semigroups} \arrow[ul, "\textit{add identities}"] \arrow[ur, "?"'] &
\end{tikzcd}
$$



Semicategories

Semigroups

Definition (Semigroup). A *semigroup* **S** is a set $\mathbf{S}$, together with a binary operation

$$⨟ : \mathbf{S} \times \mathbf{S} \to \mathbf{S},$$

called *composition*, which satisfies the *associative* law:

$$(x ⨟ y) ⨟ z = x ⨟ (y ⨟ z)$$

for all $x, y, z \in \mathbf{S}$.

Examples:

$\langle \mathbb{R}, \cdot \rangle$

$\langle \mathbb{R}, + \rangle$ $\langle \mathbb{N}, \min \rangle$ $\langle \mathbb{N}, \max \rangle$ $\langle \mathbb{R}, \max \rangle$

Semigroups

Example: states of development of a plant

$$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$$

$$T : X \to X$$

$$\begin{aligned} T(\text{sprout}) &= \text{young} \\ T(\text{young}) &= \text{mature} \\ T(\text{mature}) &= \text{old} \\ T(\text{old}) &= \text{dead} \\ T(\text{dead}) &= \text{dead} \end{aligned}$$

$$\mathbf{S} = \{T^n \mid n \in \mathbb{N}\}$$

$T ; T = T^2, \quad T^2 ; T = T^3, \quad \ldots$

$$T^n = T^4 \quad \forall\, n \geq 4$$

$$\mathbf{S} = \{T, T^2, T^3, T^4\}$$

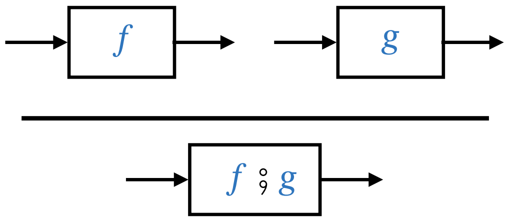
Semigroups

Example: discrete time linear time-invariant systems of the form:

$$\begin{aligned} x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k \\ y_k &= \mathbf{C}x_k \end{aligned}$$

with the constraint that input and output have the same dimension.

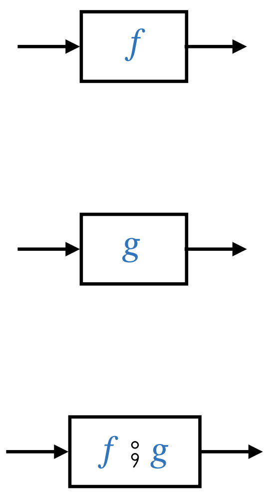
- The composition is the series composition.
- The composition of linear system is linear
  $\rightarrow$ we have a semigroup.



Semigroups

Example: discrete time linear time-invariant systems of the form:

$$
\begin{aligned}
x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k \\
y_k &= \mathbf{C}x_k
\end{aligned}
$$



Semicategories

Definition 11.6 (Semicategory). A *semicategory* $\mathbf{C}$ is:

Constituents

1. Objects: a collection[‡] $\mathrm{Ob}_{\mathbf{C}}$, whose elements are called *objects*.
2. Morphisms: for every pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, there is a set $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, elements of which are called *morphisms* from X to Y. The set is called the "hom-set from X to Y".
3. Composition operations: given any morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$ and any morphism $g \in \mathrm{Hom}_{\mathbf{C}}(Y;Z)$, there exists a morphism $f \mathbin{⨾} g \in \mathrm{Hom}_{\mathbf{C}}(X;Z)$ which is the *composition of f and g*.

Conditions

1. Associativity: for any morphisms $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$, $g \in \mathrm{Hom}_{\mathbf{C}}(Y;Z)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(Z;W)$,

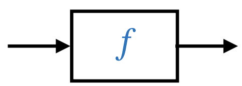
$$(f \mathbin{⨾} g) \mathbin{⨾} h = f \mathbin{⨾} (g \mathbin{⨾} h). \tag{11.1}$$

Semicategories

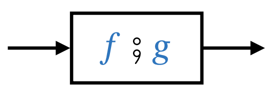
Example: discrete time linear time-invariant systems of the form:

$$
\begin{aligned}
x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k \\
y_k &= \mathbf{C}x_k
\end{aligned}
$$

- Objects:
- Morphisms:



- Composition:



- Associativity:

Semicategories

Example: states of a plant

$$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$$

$$T : X \to X$$

$$\begin{aligned} T(\text{sprout}) &= \text{young} \\ T(\text{young}) &= \text{mature} \\ T(\text{mature}) &= \text{old} \\ T(\text{old}) &= \text{dead} \\ T(\text{dead}) &= \text{dead} \end{aligned}$$

$$\mathbf{S} = \{T, T^2, T^3, T^4\}$$

$$Y = \{\text{alive}, \text{dead}\}$$

$$T' : Y \to Y$$

$$\begin{aligned} T'(\text{alive}) &= \text{dead} \\ T'(\text{dead}) &= \text{dead} \end{aligned}$$

$$\mathbf{S}' = \{T'\}$$

$$R : X \to Y$$

$$\begin{aligned} R(\text{sprout}) &= \text{alive} \\ R(\text{young}) &= \text{alive} \\ R(\text{mature}) &= \text{alive} \\ R(\text{old}) &= \text{alive} \\ R(\text{dead}) &= \text{dead} \end{aligned}$$

Semicategories

Example: states of a plant

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

$Y = \{\text{alive}, \text{dead}\}$

$T : X \to X$

$$\begin{aligned} T(\text{sprout}) &= \text{young} \\ T(\text{young}) &= \text{mature} \\ T(\text{mature}) &= \text{old} \\ T(\text{old}) &= \text{dead} \\ T(\text{dead}) &= \text{dead} \end{aligned}$$

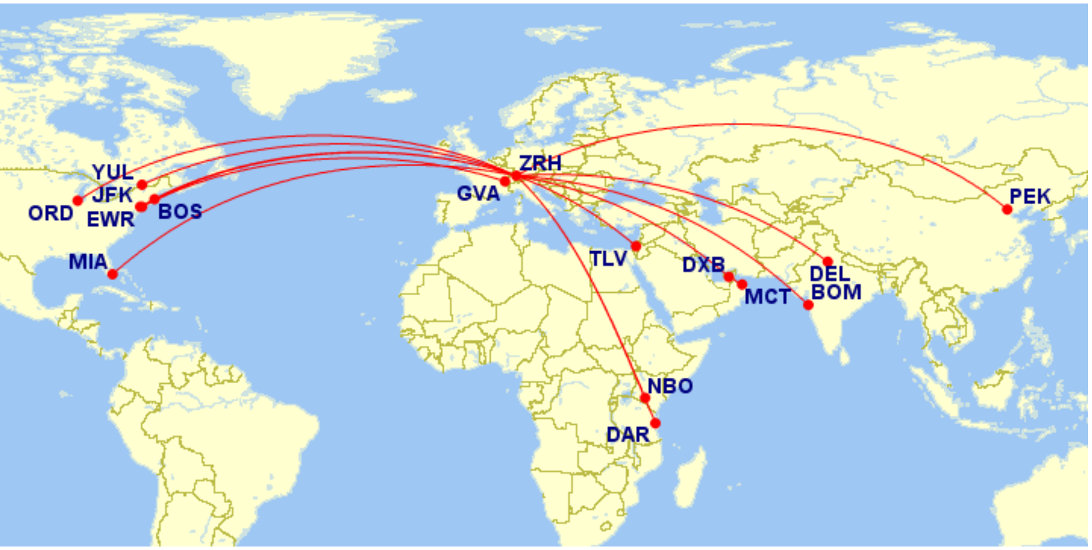
$T' : Y \to Y$

$$\begin{aligned} T'(\text{alive}) &= \text{dead} \\ T'(\text{dead}) &= \text{dead} \end{aligned}$$

$\mathbf{S} = \{T, T^2, T^3, T^4\}$

$\mathbf{S}' = \{T'\}$

$R : X \to Y$



Categories

From semigroups to monoids

A **monoid** is a semigroup with a ***neutral element***.

Definition (Monoid)**.** A *monoid* $\mathbf{M}$ is:

Constituents

1. a set $\mathbf{M}$;
2. a binary operation $⨟ : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$;

new 3. a specified element $\mathrm{id} \in \mathbf{M}$, called *neutral element*.

Conditions

1. Associative law: $(x ⨟ y) ⨟ z = x ⨟ (y ⨟ z)$;

new 2. Neutrality Laws: $\mathrm{id} ⨟ x = x = x ⨟ \mathrm{id}$.

Examples

$\langle \mathbb{R}, \cdot \rangle$

$\langle \mathbb{R}, + \rangle$ $\langle \mathbb{N}, \min \rangle$ $\langle \mathbb{N}, \max \rangle$ $\langle \mathbb{R}, \max \rangle$

From semigroups to monoids

- Sometime we can “formally add” a neutral element to a semigroup.
- Example:

$$\langle \mathbb{N}, \min \rangle$$

From semigroups to monoids

- Sometime we can "formally add" a neutral element to a semigroup.

- Example:

$$\langle \mathbb{R}, \max \rangle$$

From semigroups to monoids

- Sometime we can “formally add” a neutral element to a semigroup.

- Example:

$\mathbf{A} = \{{\color{blue}\bullet}, {\color{red}\bullet}\}$ ${\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}$

$\mathbf{S} = \{\text{set of non-empty strings of elements of } \mathbf{A}\}$

$${\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet} \fatsemi {\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet} = {\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}$$

$${\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet} \fatsemi \langle\rangle = {\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}$$

From semigroups to monoids

Example: states of development of a plant

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

$T : X \to X$

$\mathbf{S} = \{T, T^2, T^3, T^4\}$

$\mathbf{M} = \{\mathrm{id}, T, T^2, T^3, T^4\}$

From semicategories to categories

Example: states of development of a plant

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

$T : X \to X$

$\mathbf{S} = \{T, T^2, T^3, T^4\}$

$Y = \{\text{alive}, \text{dead}\}$

$T' : Y \to Y$

$\mathbf{S}' = \{T'\}$

$$R : X \to Y$$

From semicategories to categories

Example: states of development of a plant

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

$Y = \{\text{alive}, \text{dead}\}$

$T : X \to X$

$T' : Y \to Y$

$\mathbf{M} = \{\text{id}, T, T^2, T^3, T^4\}$

$\mathbf{M}' = \{\text{id}, T'\}$

$$R : X \to Y$$

Categories

Definition 1.10 (Category). A *category* $\mathbf{C}$ is:

Constituents

1. Objects: a collection $\mathrm{Ob}_{\mathbf{C}}$, whose elements are called *objects*.
2. Morphisms: for every pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, there is a set $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, elements of which are called *morphisms* from X to Y. The set is called the "hom-set from X to Y".

new 3. Identity morphisms: for each object X, there is an element $\mathrm{Id}_X \in \mathrm{Hom}_{\mathbf{C}}(X;X)$ which is called *the identity morphism of* X.

4. Composition operations: given any morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$ and any morphism $g \in \mathrm{Hom}_{\mathbf{C}}(Y;Z)$, there exists a morphism $f \,\mathbin{;}\, g \in \mathrm{Hom}_{\mathbf{C}}(X;Z)$ which is the *composition of* f *and* g.

Conditions

new 1. Unitality: for any morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$,

$$\mathrm{Id}_X \mathbin{;} f = f = f \mathbin{;} \mathrm{Id}_Y . \tag{1.5}$$

2. Associativity: for any morphisms $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$, $g \in \mathrm{Hom}_{\mathbf{C}}(Y;Z)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(Z;W)$,

$$(f \mathbin{;} g) \mathbin{;} h = f \mathbin{;} (g \mathbin{;} h). \tag{1.6}$$

Categories

Example: states of development of a plant

$X = \{\text{sprout}, \text{young}, \text{mature}, \text{old}, \text{dead}\}$

$Y = \{\text{alive}, \text{dead}\}$

$T : X \to X$

$T' : Y \to Y$

$\mathbf{M} = \{\text{id}, T, T^2, T^3, T^4\}$

$\mathbf{M}' = \{\text{id}, T'\}$

$$R : X \to Y$$

Categories

Example: sets and functions

Categories

Example: *finite* sets and functions

Example: *a specified collection of* sets and functions

Categories

Example: matrices over the real numbers

Categories

Example: finite-dimensional vector spaces over the real numbers

Example: all vector spaces over the real numbers

Categories

Example: monoids and their morphisms

A monoid as a category

Given a monoid

$$\langle \mathbf{M}, ⨟, \mathrm{id} \rangle$$

we can think of it as a special case of a category.

Level shifts

A specific monoid, viewed as a category

$$\langle \mathbf{M}, \fatsemi, \mathrm{id} \rangle$$

The category of all monoids

Summary so far

$$\begin{tikzcd}
 & \textit{Categories} & \\
\textit{Monoids} \arrow[ur, "\text{add “objects”}"] & & \textit{Semicategories} \arrow[ul, "\text{add identities}"'] \\
 & \textit{Semigroups} \arrow[ul, "\text{add identities}"] \arrow[ur, "\text{add “objects”}"'] &
\end{tikzcd}$$