Applied Compositional Thinking for Engineers

Spring 2021

Monads
(continued)

Another perspective on Monads

- Kleisli perspective:
  - Monads can be used to model “generalized objects” and “generalized morphisms”
  - Examples: powersets, probabilities, intervals, functions with side-effects, ...
  - Basic idea:
- M-algebras perspective:
  - Monads can be used to “encode theories of operations” or to “blueprint algebraic gadgets”
  - Examples: the theory of monoids, monoid actions, ...
  - Basic idea:

Formal expressions

- What is a “formal expression” ?

X = {🍎, 🍌, 🥕}   🍎🍌🥕

Formal expressions

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$X = \{$🍎, 🍌, 🥕$\}$ 🍎🍌🥕

[🍎]

[🍌]

[🥕]

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🍎🍌🥕

[🍎]

[🍎] $*$ [🍌]

[🍌]

[🥕]

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[🍎]   [🍎] ∗ [🍌]   [[🍎] ∗ [🍌]]

[🍌]

[🥕]   [🍎] ∗ [🍌] ∗ [🥕]   [[🍎] ∗ [🍌] ∗ [🥕]]

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Evaluating formal expressions

- What do we mean by “evaluating” ?

[🍎] ∗ [🍌] $\xrightarrow{?}$ 🍎 + 🍌

[🍎] ∗ [🍌] ∗ [🥕] $\xrightarrow{?}$ 🍎 + 🍌 + 🥕

[🍎] ∗ [🍎] ∗ [🍎] $\xrightarrow{?}$ 🍎 + 🍎 + 🍎

[[🍎] ∗ [🍌]] $\xrightarrow{?}$ [🍎] ∗ [🍌]

[[🍎] ∗ [🍌]] ∗ [[🍎] ∗ [🍎] ∗ [🥕]] ∗ [[🍎]] $\xrightarrow{?}$

Monad for formal expressions

- Suppose we have a monad that maps a set to the set of formal multiplications of its elements:

X = {🍎, 🍌, 🥕} ⇝ MX = { [ ], [🍎], [🍌], [🥕], [🍎] * [🍌], … "all finite multiplications of formal elements of X" }

Monad for formal expressions

- Suppose we have a monad that maps a set to the set of formal multiplications of its elements:

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Evaluating / calculating formal multiplications can be encoded using the notion of an <u>M-algebra</u>

$$MX \xrightarrow{a} X$$

[🍎] * [🍌] ⟶ 🍎 + 🍌

Algebras for a monad

Definition (Algebra of a monad)**.** Let $\langle M, \mathrm{un}, \mathrm{mu} \rangle$ be a monad on a category $\mathbf{C}$. An algebra of M (also called an M-algebra) is specified by:

<u>Constituents</u>

1. an object X of $\mathbf{C}$;
2. a morphism $a : M(X) \to X$ of $\mathbf{C}$.

<u>Conditions</u>

1. *Unit*: the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{un}_X"] \arrow[dr, "\mathrm{Id}"'] & M(X) \arrow[d, "a"] \\ & X \end{tikzcd}$$

   commutes.
2. *Composition*: the diagram

$$\begin{tikzcd} (M \mathbin{;} M)(X) \arrow[r, "Ma"] \arrow[d, "\mathrm{mu}_X"'] & M(X) \arrow[d, "a"] \\ M(X) \arrow[r, "a"'] & X \end{tikzcd}$$

   commutes

Blueprinting monoids

MX = "formal products of elements of X"

$MX \xrightarrow{a} X$ "an operation for evaluating formal products"

Unit of the monad $\quad un_X : X \to MX$

$x \longmapsto [x]$ "formal product of just "x" alone"

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$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "un_X"'] & Y \arrow[d, "un_X"] \\ MX \arrow[r, "Mf"'] & MY \end{tikzcd}$$

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MX \arrow[r, "Mf"'] & MY
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & fx \arrow[d, mapsto] \\
{[x]} \arrow[r, mapsto] & {[fx]}
\end{tikzcd}$$

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Multiplication of the monad $\quad mu_X : MMX \longrightarrow MX$

$$[[x_1] * [x_2]] * [[x_3]] \longmapsto [x_1] * [x_2] * [x_3]$$

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"remove the outer brackets"

$$\begin{tikzcd}
MMMX \arrow[r, "M mu_X"] \arrow[d, "mu_{MX}"'] & MMX \arrow[d, "mu_X"] \\
MMX \arrow[r, "mu_X"'] & MX
\end{tikzcd}$$

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MMX \arrow[r, "mu_X"'] & MX
\end{tikzcd}$$

$$\begin{tikzcd}
{[[[x_1]*[x_2]]*[[x_3]]]*[[[x_4]]]} \arrow[r, mapsto] \arrow[d, mapsto] & {[[x_1]*[x_3]]*[[x_3]]*[[x_4]]} \arrow[d, mapsto] \\
{[[x_1]*[x_2]*[x_3]]*[[x_4]]} \arrow[r, mapsto] & {[x_1]*[x_2]*[x_3]*[x_4]}
\end{tikzcd}$$

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$X = \{1, 2, 3, \dots\}$

$MX = \{[\ ], [1], [2], \dots [[0] * [1]], \dots, [[1] * [2] * [3]], \dots\}$

Let's define "$*$" to mean "addition"

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Let's define "$*$" to mean "addition"

$MX \xrightarrow[a]{} X$ "addition map"

$[[0] * [1]] \longmapsto 1 \quad (= "0 + 1")$

$[[1] * [2] * [3]] \longmapsto 6 \quad (= "1 + 2 + 3")$

Blueprinting monoids

$$MX \xrightarrow{a} X$$

<u>Unit Law</u>

$$\begin{tikzcd} X \arrow[r, "mu_X"] \arrow[dr, "Id"'] & MX \arrow[d, "a"] \\ & X \end{tikzcd}$$

Blueprinting monoids

$$MX \xrightarrow{a} X$$

Unit Law

$$\begin{tikzcd}
X \arrow[r, "mu_X"] \arrow[dr, "Id"'] & MX \arrow[d, "a"] \\
& X
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto, "mu"] \arrow[dr, mapsto, "Id"'] & {[x]} \arrow[d, mapsto, "a"] \\
& x
\end{tikzcd}$$

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Unit Law

$$\begin{tikzcd}
X \arrow[r, "mu_X"] \arrow[dr, "Id"'] & MX \arrow[d, "a"] \\
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\qquad
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x \arrow[r, mapsto, "mu"] \arrow[dr, mapsto, "Id"'] & {[x]} \arrow[d, mapsto, "a"] \\
& x
\end{tikzcd}$$

Composition Law

$$\begin{tikzcd}
MMX \arrow[r, "Ma"] \arrow[d, "mu_X"'] & MX \arrow[d, "a"] \\
MX \arrow[r, "a"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
{[[1]*[2]]*[[3]]} \arrow[r, mapsto] \arrow[d, mapsto] & {[a([1]*[2])]*[a([3])]} \\
& {= [3]*[3]} \arrow[d, mapsto, "a"] \\
{[1]*[2]*[3]} \arrow[r, mapsto, "a"'] & 6
\end{tikzcd}$$

Blueprinting monoids

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Proposition:

M-algebra structure on X $\iff$ X is equipped with a monoid structure

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M-algebra structure on X $\iff$ X is equipped with a monoid structure

- the neutral element of X corresponds to $a([\,])$

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Proposition:

M-algebra structure on X $\iff$ X is equipped with a monoid structure

- the neutral element of X corresponds to $a([\,])$
- the composition in X corresponds to the action of a

$$[x_1]*[x_2] \overset{a}{\longmapsto} x_1 x_2$$

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Proposition:

M-algebra structure on X $\iff$ X is equipped with a monoid structure

- the neutral element of X corresponds to $a([\ ])$
- the composition in X corresponds to the action of a

$$[x_1] * [x_2] \overset{a}{\longmapsto} x_1 x_2$$

- associativity, unitality of X encoded by M-algebra axioms

Actions as M-algebras

X a set

A a monoid

M the "writer monad"

$MX = X \times A$

Actions as M-algebras

X a set

A a monoid

M the "writer monad"

$$MX = X \times A$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \quad \overset{M}{\rightsquigarrow} \quad \begin{tikzcd} X \times A \arrow[d, "f \times \mathrm{Id}_A"] \\ Y \times A \end{tikzcd}$$

Actions as M-algebras

$$
\begin{tikzcd}
MX \arrow[d, "a"'] \\
X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \times A \arrow[d, "a"] \\
X
\end{tikzcd}
$$

Unit Law:

$$
\begin{tikzcd}
X \arrow[r, "un_X"] \arrow[dr, "Id_X"'] & X \times A \arrow[d, "a"] \\
 & X
\end{tikzcd}
$$

Actions as M-algebras

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Unit Law:

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$$(un_X ; a)(x) = a(un_X(x)) = a((x, 1_A)) \overset{?}{=} x$$

Actions as M-algebras

$$\begin{tikzcd} MX \arrow[d, "a"'] \\ X \end{tikzcd} \qquad \begin{tikzcd} X \times A \arrow[d, "a"] \\ X \end{tikzcd}$$

composition law:

$$\begin{tikzcd} (X \times A) \times A \arrow[r, "Ma"] \arrow[d, "mu\,X"'] & X \times A \arrow[d, "a"] \\ X \times A \arrow[r, "a"'] & X \end{tikzcd}$$

Actions as M-algebras

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MX \arrow[d, "a"'] \\
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$$

$$(Ma\,;\,a)((x,m),m') = a\big((a \times Id_A)((x,m),m')\big) = a\big(a(x,m),m'\big)$$

Actions as M-algebras

$$
\begin{array}{ccc}
MX & \qquad & X \times A \\
\Big\downarrow a & & \Big\downarrow a \\
X & & X
\end{array}
$$

composition law:

$$
\begin{array}{ccc}
(X \times A) \times A & \xrightarrow{\;Ma\;} & X \times A \\
{\scriptstyle mu_X}\Big\downarrow & & \Big\downarrow {\scriptstyle a} \\
X \times A & \xrightarrow[\;a\;]{} & X
\end{array}
$$

$$(Ma ; a)((x, m), m') = a\big((a \times Id_A)((x,m),m')\big) = a\big(a(x,m), m'\big)$$

$$(mu_X ; a)((x,m),m') = a\big(mu_X((x,m),m')\big) = a\big((x, mm')\big)$$

Morphisms of M-algebras

Definition (M-algebra morphism)**.** Let $\langle M, \mathrm{un}, \mathrm{mu}\rangle$ be a monad on a category $\mathbf{C}$, and let $\langle X_1, a_1\rangle$ and $\langle X_2, a_2\rangle$ be algebras of M. A morphism $\langle X_1, a_1\rangle \to \langle X_2, a_2\rangle$ of M-algebras is specified by:

<u>Constituents</u>

1. A morphism $f : X_1 \to X_2$ in $\mathbf{C}$.

<u>Conditions</u>

1. The diagram

$$\begin{tikzcd} M(X_1) \arrow[r, "Mf"] \arrow[d, "a_1"'] & M(X_2) \arrow[d, "a_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

   commutes.

Morphisms of M-algebras

$$MX = X \times A$$

$$\begin{tikzcd} X \times A \arrow[r, "f \times Id_A"] \arrow[d, "a_X"'] & Y \times A \arrow[d, "a_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Morphisms of M-algebras

MX = "formal multiplications of elements of X" (List monad)

$$\begin{tikzcd} MX \arrow[r, "Mf"] \arrow[d, "a_X"'] & MY \arrow[d, "a_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\longrightarrow$ encodes morphisms of monoids

Category of M-algebras

Definition (Category of M-algebras)**.** Let $\langle M, \mathrm{un}, \mathrm{mu}\rangle$ be a monad on a category $\mathbf{C}$. The *category of M-algebras* $\mathbf{C}^M$ of the monad M is specified by:

1. *Objects*: M-algebras;
2. *Morphisms*: M-algebra morphisms;
3. *Identities*: given an M-algebra $\langle X, a\rangle$, its identity morphism is Id_X;
4. *Composition*: is induced by the composition of morphisms in $\mathbf{C}$.

- Examples:
  - List monad: category of algebras is the category of monoids
  - Writer monad: category of algebras is the category of monoid actions
  - Powerset monad: category of algebras is the category of complete semilattices

Comparing the two perspectives

Definition (Kleisli morphisms)**.** Let $\langle M, \mathrm{un}, \mathrm{mu} \rangle$ be a monad on a category $\mathbf{C}$, and let $X, Y \in \mathrm{Ob}_{\mathbf{C}}$. A *Kleisli morphism* $X \to Y$ is morphism of $\mathbf{C}$ of the form $X \to MY$.

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Definition (Kleisli composition)**.** Let $\langle M, \mathrm{un}, \mathrm{mu}\rangle$ be a monad on a category $\mathbf{C}$, let $X, Y, Z \in \mathrm{Ob}_{\mathbf{C}}$, and let $f : X \to MY$ and $g : Y \to MZ$ be morphisms in $\mathbf{C}$ (so, they are Kleisli morphisms). Their *Kleisli composition* is the morphism in $\mathbf{C}$ given by the composition

$$X \xrightarrow{f} M(Y) \xrightarrow{Mg} (M \mathbin{⨾} M)(Z) \xrightarrow{\mathrm{mu}_Z} M(Z).$$

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1. *Objects*: $\mathrm{Ob}(\mathbf{C}_M) := \mathrm{Ob}(\mathbf{C})$;
2. *Morphisms:* $\mathrm{Hom}_{\mathbf{C}_M}(X, Y) := \mathrm{Hom}_{\mathbf{C}}(X, M(Y))$;
3. *Identities*: $\mathrm{Id}_X := \mathrm{un}_X$;
4. *Composition*: Kleisli composition.

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Constituents

1. an object X of $\mathbf{C}$;
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Conditions

1. *Unit*: the diagram

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Comparing the two perspectives

Monad	*Kleisli*	*Algebras*
List	Lists generalize elements	Monoids
Writer	Accounting for side effects	Monoid actions
Powerset	Subsets generalize elements	Complete semilattices
Probability	Distributions generalize elements	Convex spaces
Interval	Intervals generalize elements	?

... and more...