Applied Compositional Thinking for Engineers

Spring 2021

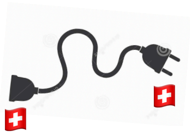
Operads

Plan

- What is an operad? The idea and examples
- Formal definition
- More examples

The basic idea

Monoids
(Category with one object)



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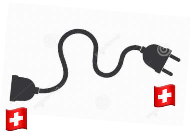
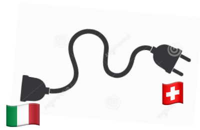
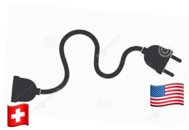
 

Categories

The basic idea

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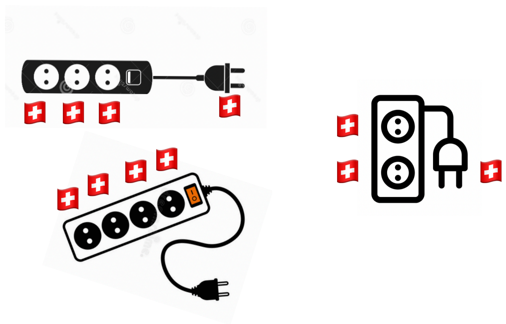
Categories

Operad with one object



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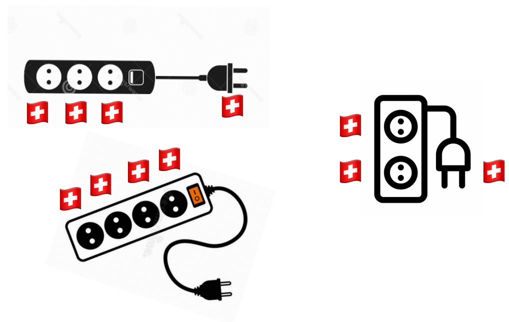
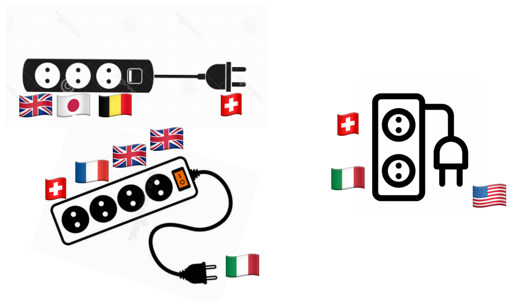
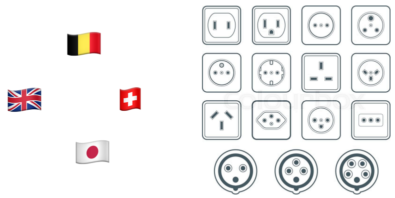
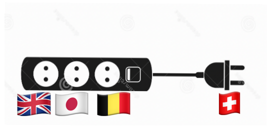
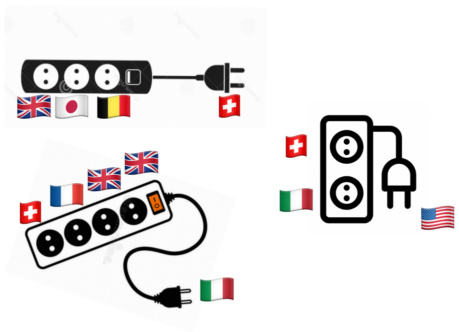


Operads (Multicategories)

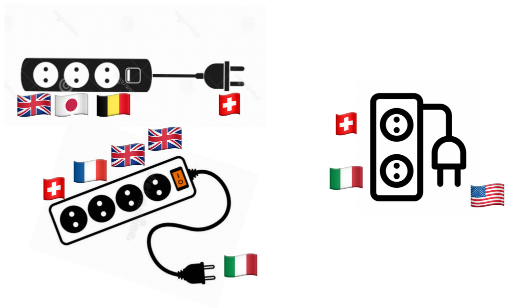


Terminology

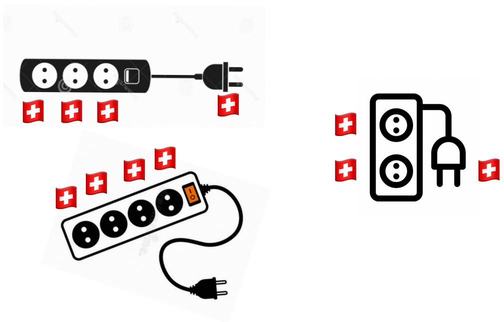
Objects	Sorts Types Things	
Morphisms	Operations Arrangements	
Composition	Nesting Composition formula	

Terminology

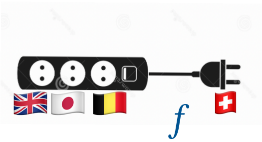
Operad	*Multicategory*	*Colored operad*	*Typed operad*

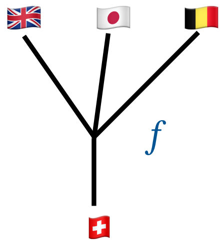


Operad with one object	*Operad*	*Untyped operad*	*Single-typed operad*



Composition





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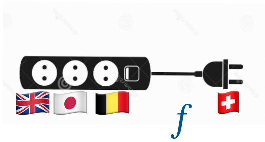

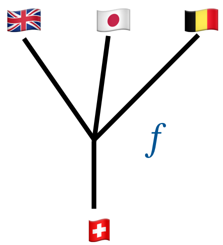

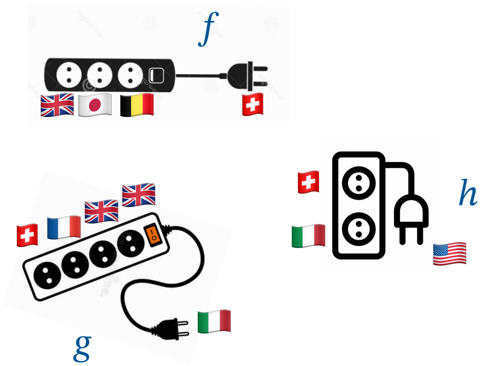

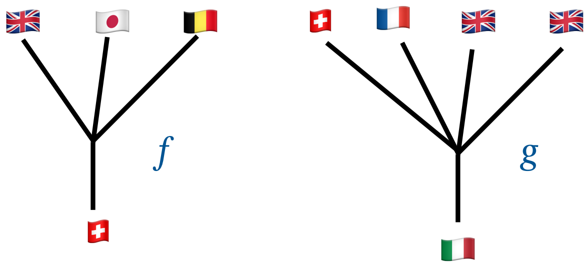

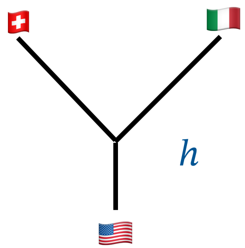

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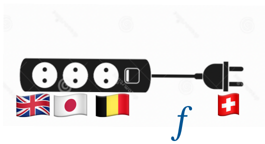

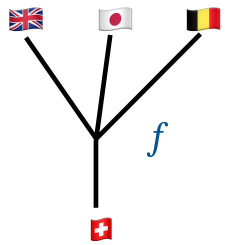

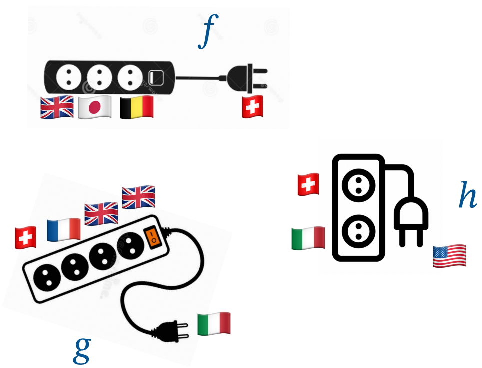

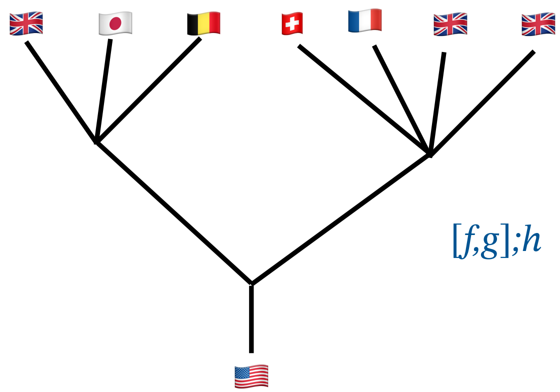

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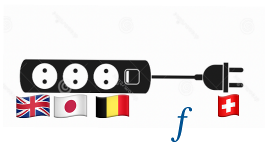
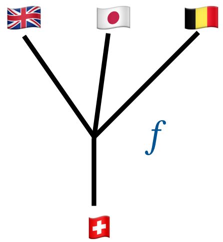
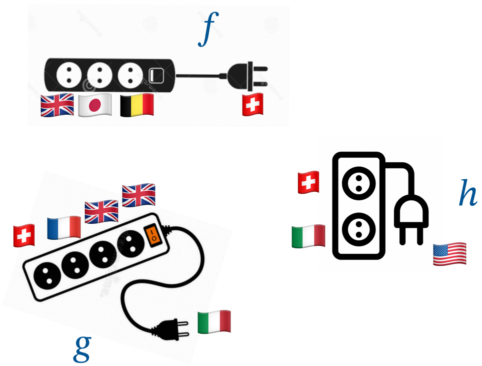
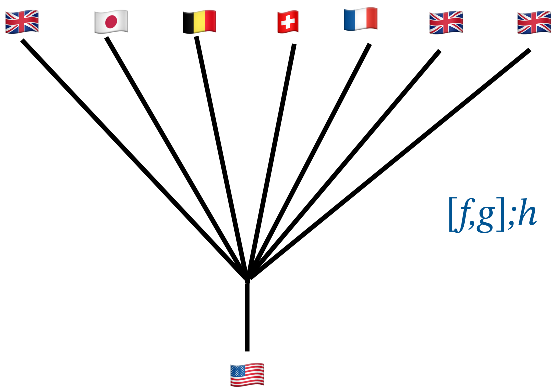
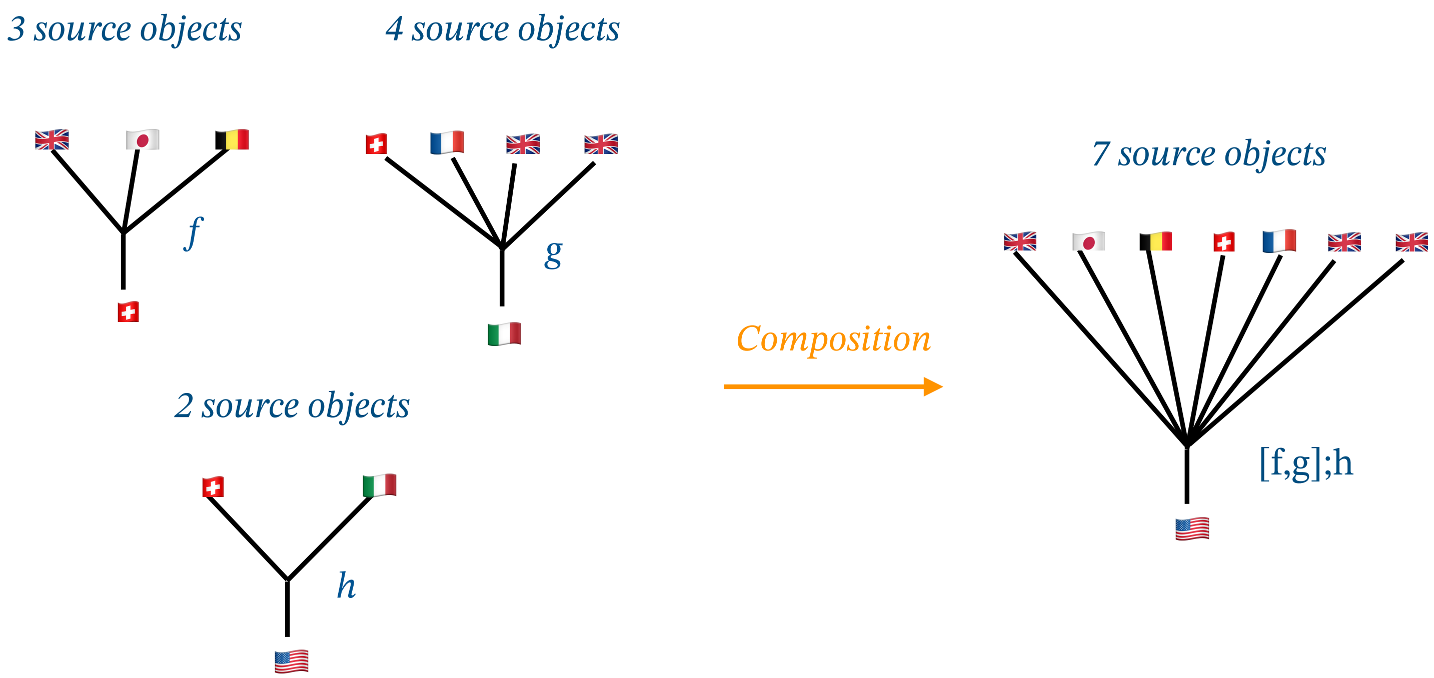






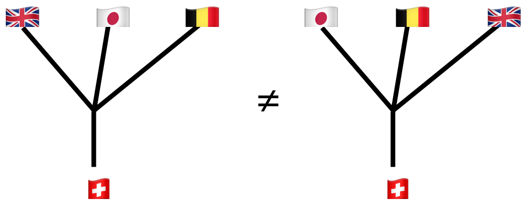


Composition

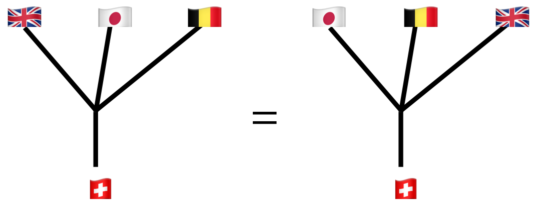


Symmetric vs unsymmetric

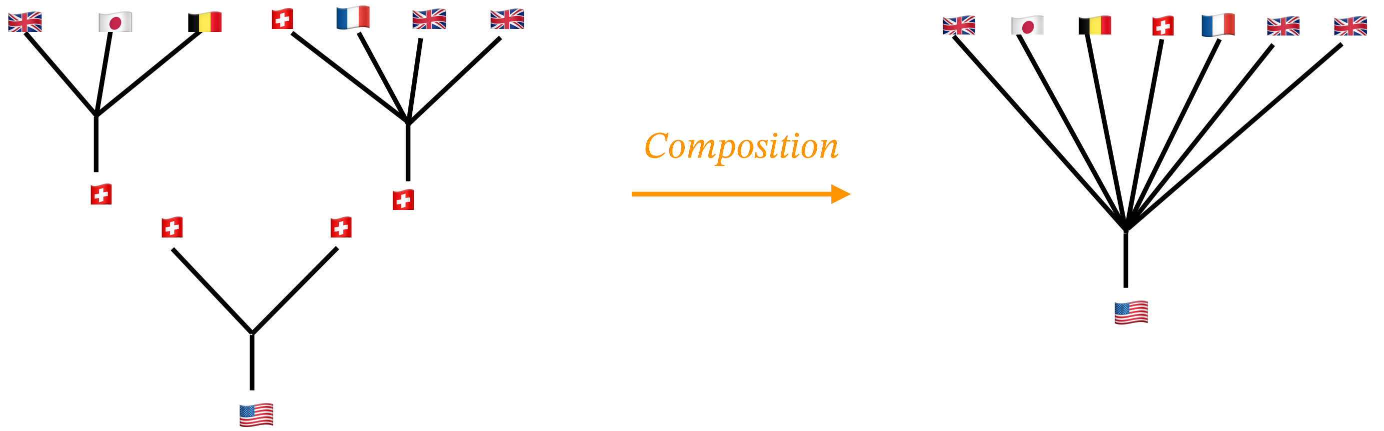
unsymmetric

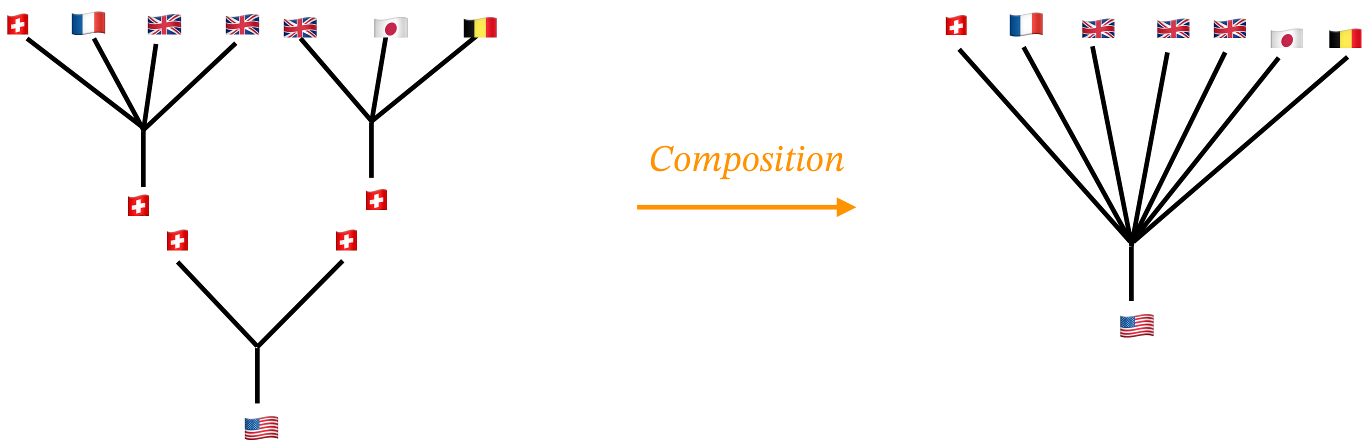


symmetric



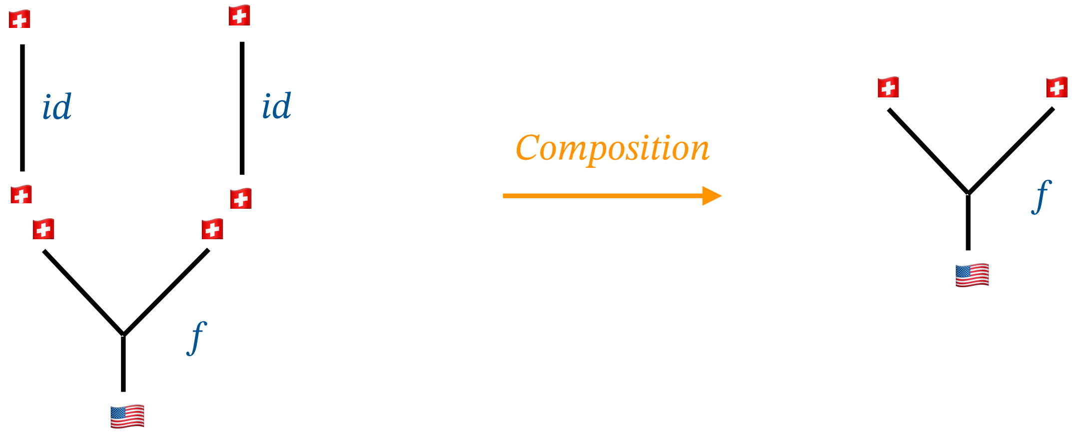
Composition and symmetry





Identity morphisms

Identity morphisms



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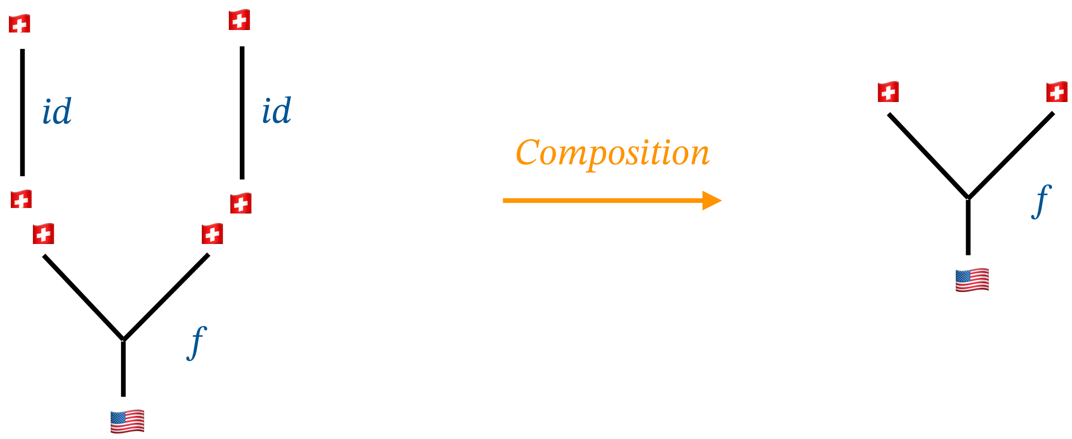
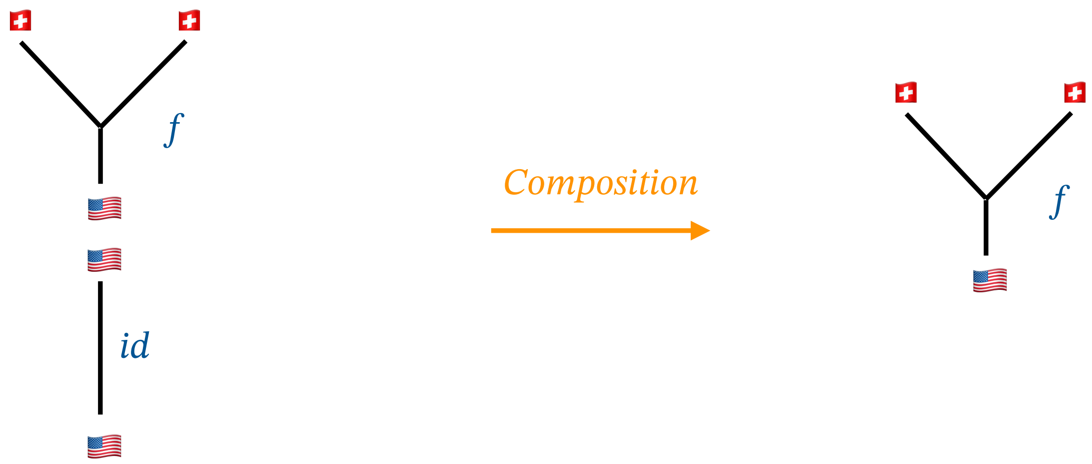
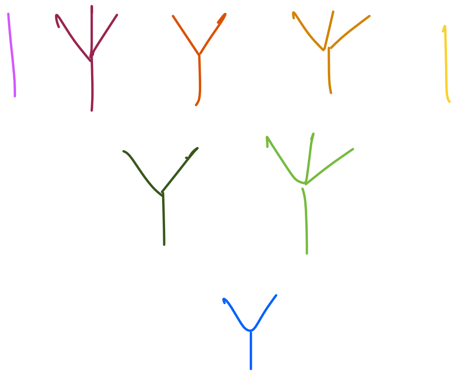




Composition is associative



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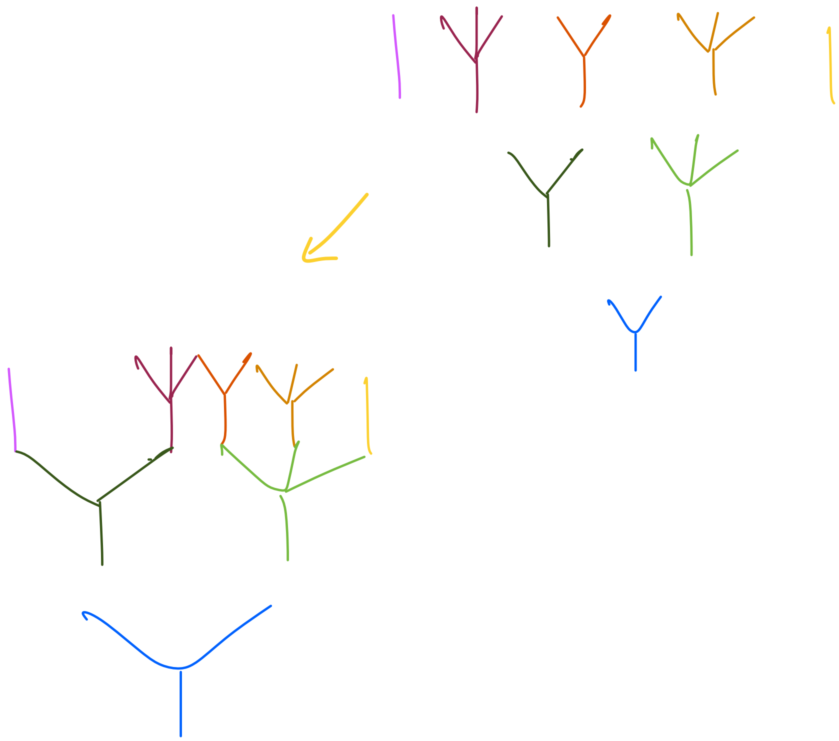

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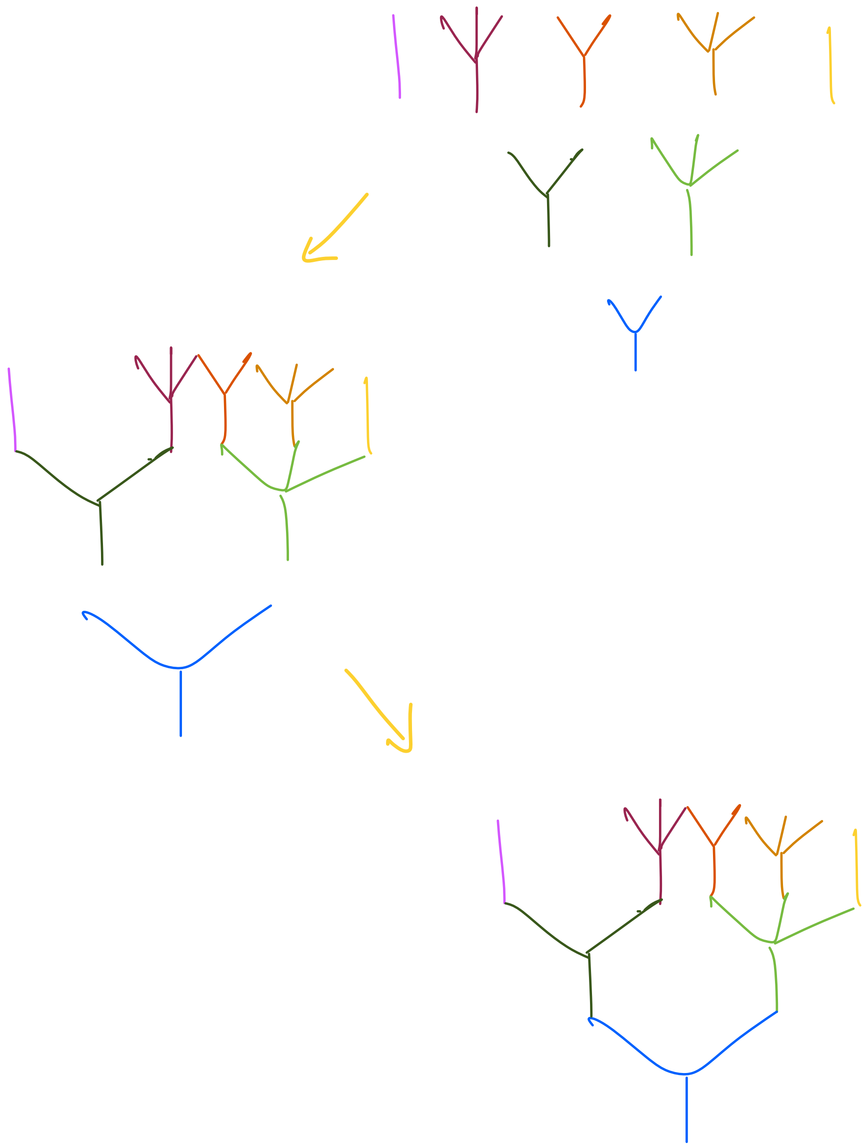

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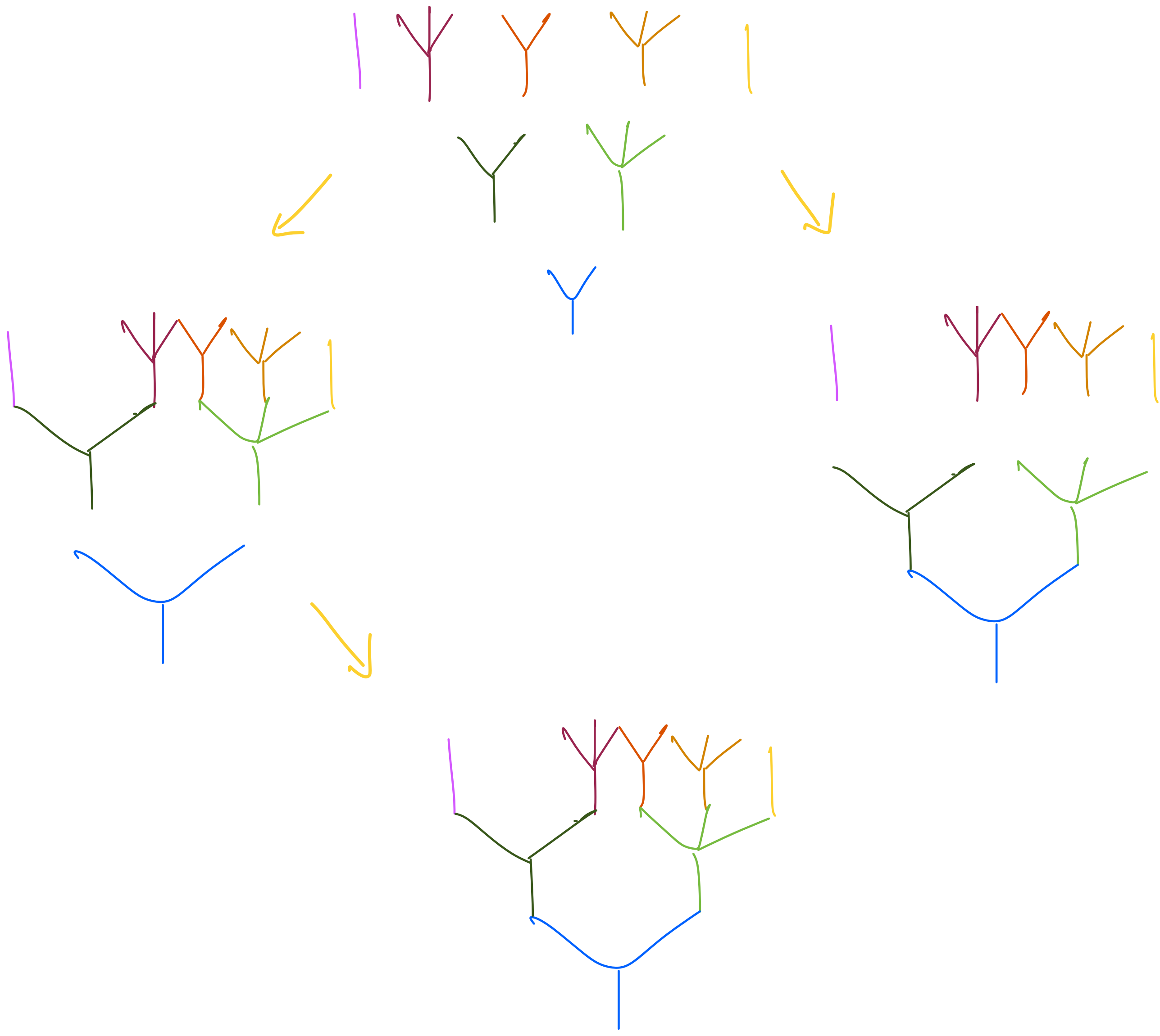

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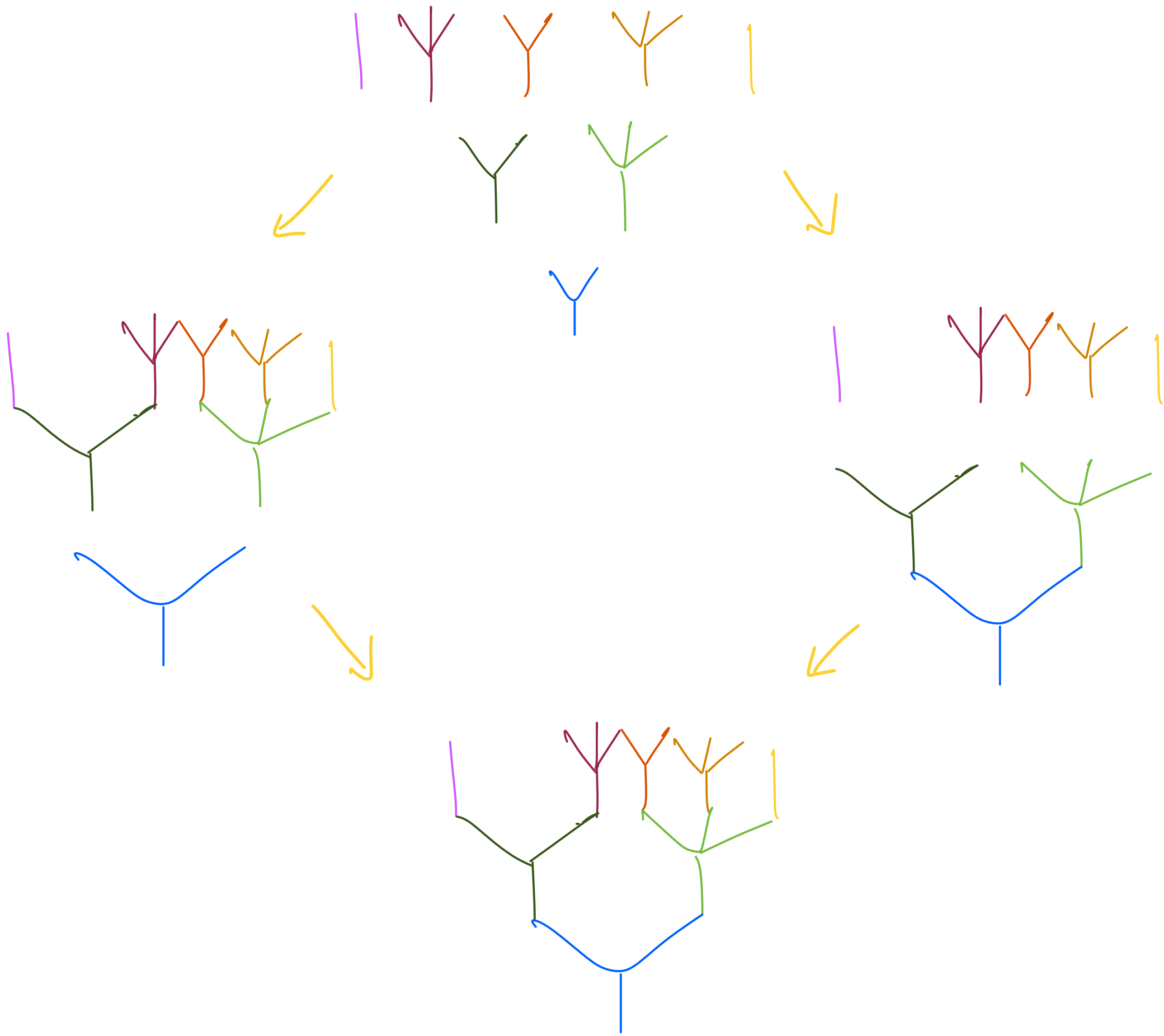
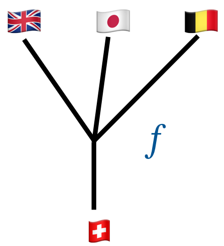
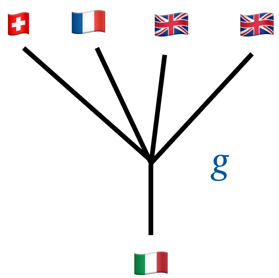


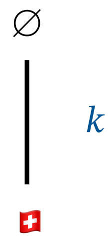
Notation



[🇬🇧,🇯🇵,🇧🇪] $\xrightarrow{f}$ 🇨🇭



[🇨🇭,🇫🇷,🇬🇧,🇬🇧] $\xrightarrow{g}$ 🇮🇹



[] $\xrightarrow{k}$ 🇨🇭

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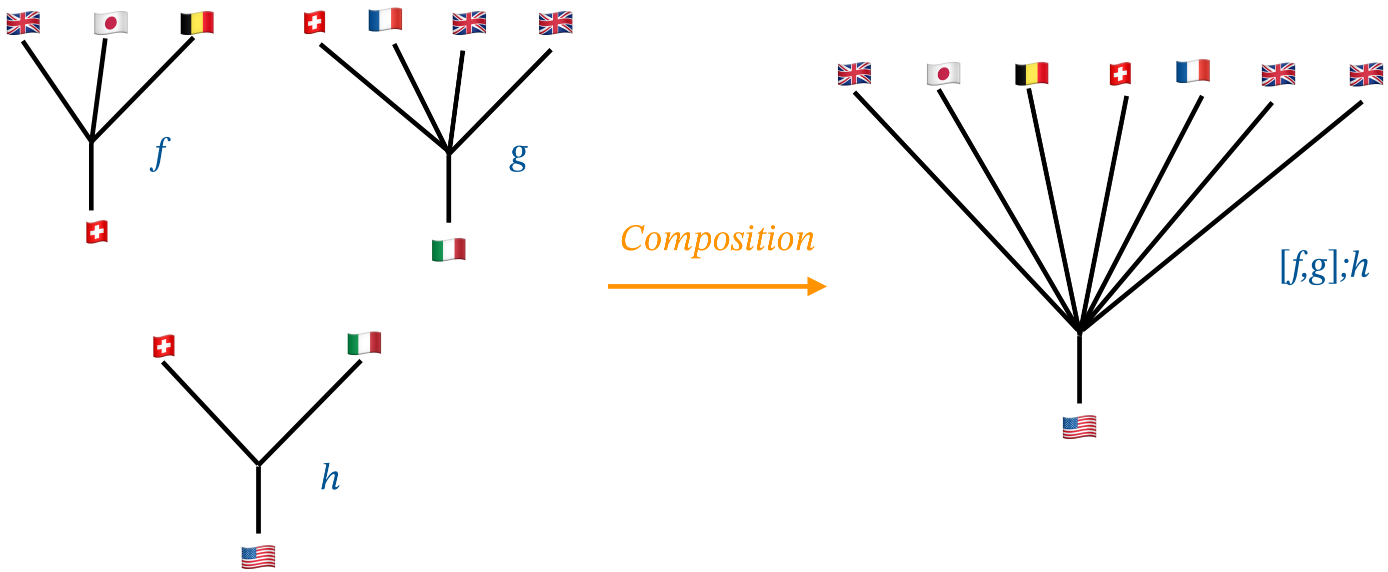

Operad of sets (via product)

$\mathcal{O} = \mathrm{Set}$ $\big(\longleftarrow$ fix as a standard notation $\big)$

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$\mathcal{O} = Set$

Objects $ob(Set)$ = all sets

Operad of sets (via product)

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Objects $\quad ob(Set) = \text{all sets}$

Morphisms $\quad Set([X_1, \dots, X_n]; Y) = Hom_{Set}(X_1 \times \cdots \times X_n; Y)$

$$= \text{functions } X_1 \times \cdots \times X_n \longrightarrow Y$$

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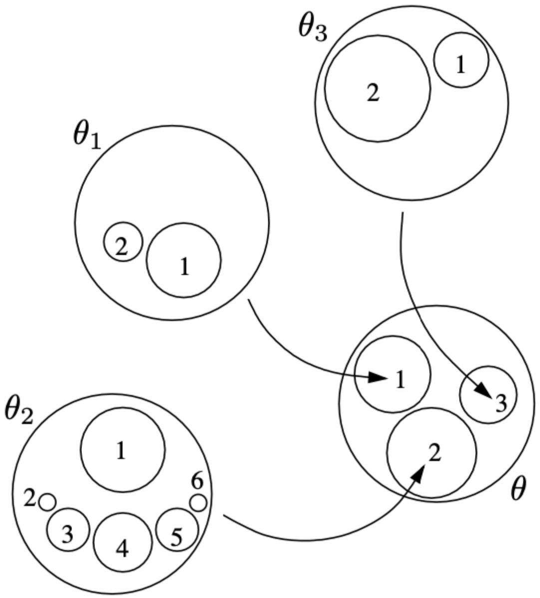
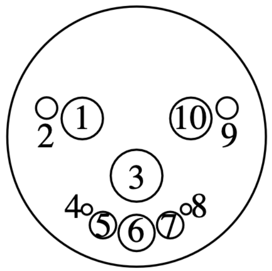
Composition

$$X_1 \times X_2 \times X_3 \xrightarrow{f} Y_1 \qquad X_4 \times X_5 \xrightarrow{g} Y_2 \qquad Y_1 \times Y_2 \xrightarrow{h} Z \qquad \rightsquigarrow \qquad X_1 \times X_2 \times X_3 \times X_4 \times X_5 \xrightarrow{(f \times g);h} Z$$

Example: Operad of little disks





Operads: formal definition

The <u>constituents</u> to define an operad $\mathcal{O}$ are:

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$$\mathcal{O}([x_1, \dots, x_n]; y) \qquad \text{"morphisms } [x_1, \dots, x_n] \to y\text{"}$$

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Composition : functions

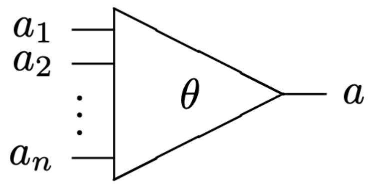
$$\mathcal{O}([x_1^1, \dots, x_{n_1}^1], y_1) \times \dots \times \mathcal{O}([x_1^m, \dots, x_{n_m}^m], y_m) \times \mathcal{O}([y_1, \dots, y_m]; z)$$

$$\langle f_1, \dots, f_m, g \rangle$$

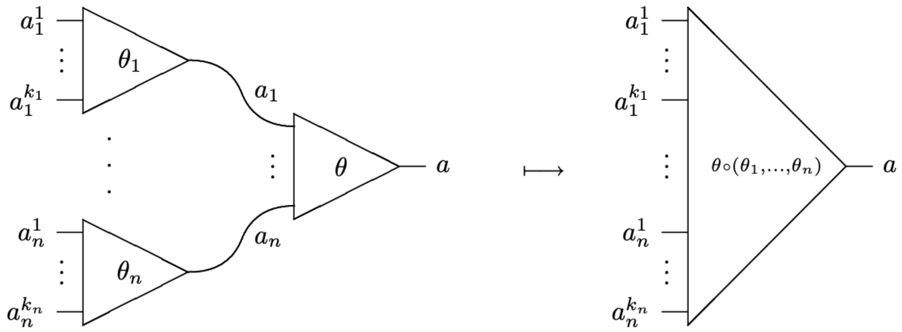
$$\mathcal{O}([x_1^1, \dots, x_{n_m}^m]; z)$$

$$[f_1, \dots, f_m]; g$$

Operads: pictures



Operads: pictures



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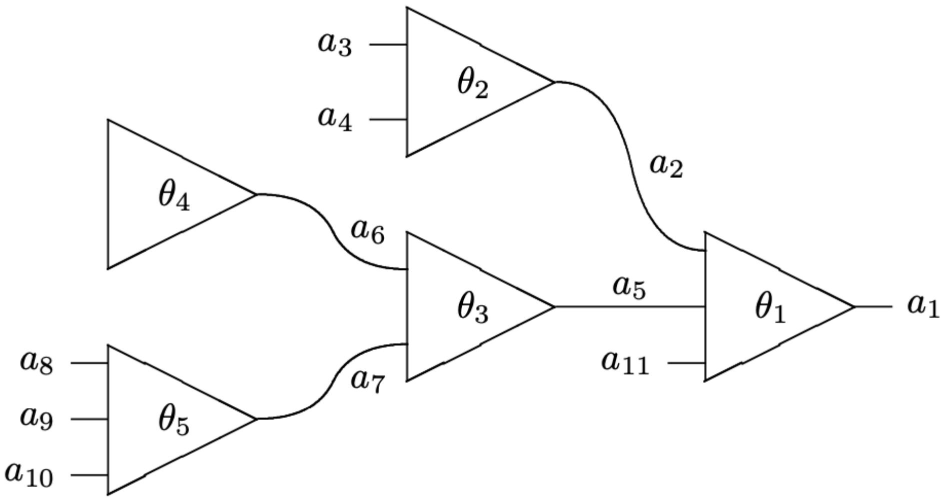


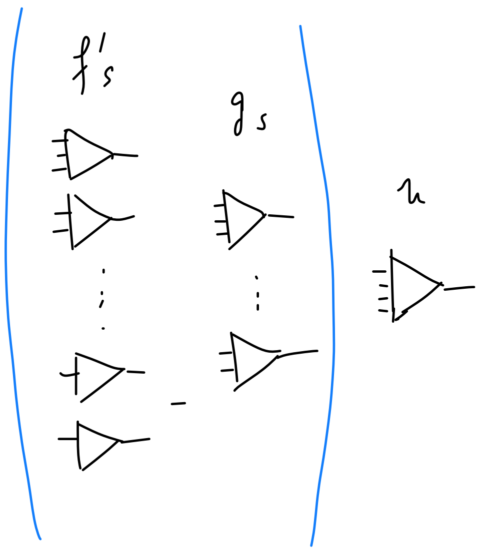
Operads: formal definition

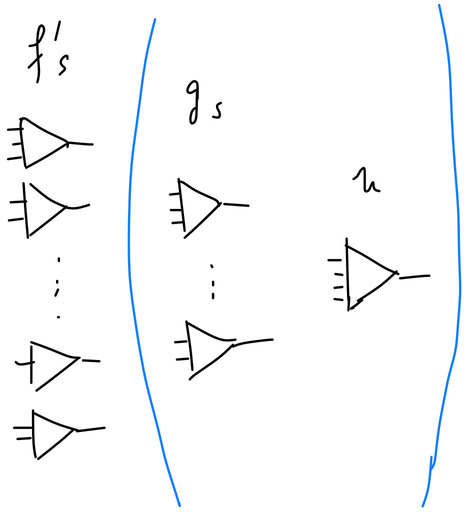
Associative law

$$[[f^1_1,\dots,f^1_{n_1}]; g_1, [f^2_1,\dots,f^2_{n_2}]; g_2, \dots, [f^m_1,\dots,f^m_{n_m}]; g_m]; h$$

$$=$$

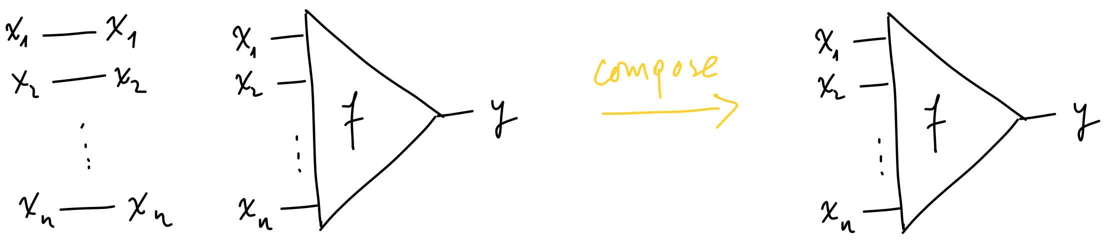
$$[f^1_1,\dots,f^m_{n_m}]; ([g_1,\dots,g_m]; h)$$

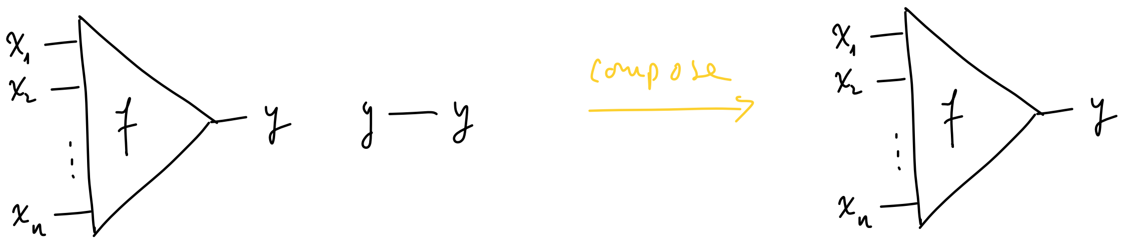




Operads: formal definition

Unit laws $[id_{x_1}, \dots, id_{x_n}] ; f = f = f ; id_y \quad \forall f : [x_1, \dots, x_n] \to y$





Operad of multilinear maps

Objects finite dim'l $\mathbb{R}$-vector spaces

Morphisms $\mathcal{O}([V_1, \dots, V_n]; W)$ = multilinear maps $V_1 \times \dots \times V_n \longrightarrow W$

$\mathcal{O}([\,]; W)$ = linear maps $\mathbb{R} \to W$

Composition usual composition

Operad from a moidal category

Let $\langle \mathcal{C}, \otimes, 1 \rangle$ be a strict monoidal category. Define its associated operad $\mathcal{O}_{\mathcal{C}}$ as follows:

Objects $\quad Ob(\mathcal{O}_{\mathcal{C}}) = Ob(\mathcal{C})$

Morphisms $\quad \mathcal{O}_{\mathcal{C}}([x_1, \dots, x_n]; y) = Hom_{\mathcal{C}}(x_1 \otimes \cdots \otimes x_n ; y)$

Composition

$$\begin{array}{c} x_1 \\ \otimes \\ x_2 \end{array} \xrightarrow{f} y_1 \qquad \begin{array}{c} x_3 \\ \otimes \\ x_4 \\ \otimes \\ x_5 \end{array} \xrightarrow{g} y_2 \qquad \begin{array}{c} y_1 \\ \otimes \\ y_2 \end{array} \xrightarrow{h} z \quad \rightsquigarrow \quad \begin{array}{c} x_1 \\ \otimes \\ x_2 \\ \otimes \\ x_3 \\ \otimes \\ x_4 \\ \otimes \\ x_5 \end{array} \xrightarrow{f \otimes g \,;\, h} z$$

Operad from a moidal category

Morphisms $\mathcal{O}_{\mathcal{C}}([x_1, \dots, x_n]; y) = Hom_{\mathcal{C}}(x_1 \otimes \cdots \otimes x_n; y)$

- $\mathcal{O}_{\mathcal{C}}([\,], y) = Hom_{\mathcal{C}}(1, y)$

- id_x in $\mathcal{O}_{\mathcal{C}}([x]; x)$ is id_x in $Hom_{\mathcal{C}}(x; x)$

Operad of sets (via product)

$\mathcal{O} = \mathsf{Set}$

Objects $\quad ob(\mathsf{Set}) = $ all sets

Morphisms $\quad \mathsf{Set}([X_1, \dots, X_n]; Y) = \mathrm{Hom}_{\mathsf{Set}}(X_1 \times \cdots \times X_n; Y)$

$\qquad = $ functions $X_1 \times \cdots \times X_n \longrightarrow Y$

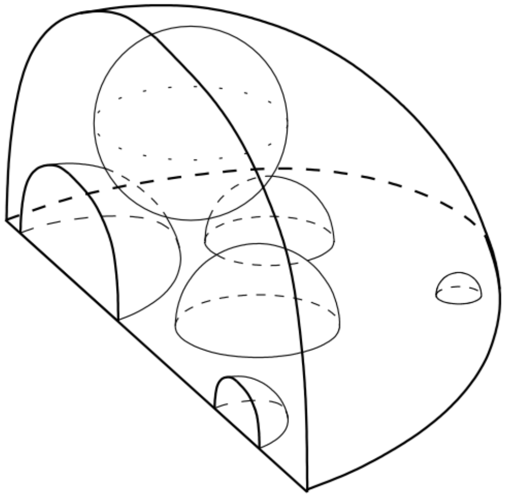
- $\mathsf{Set}([\,]; Y) = \mathrm{Hom}_{\mathsf{Set}}(\{*\}; Y) = $ functions $\{*\} \to Y$ "elements of Y"
- $id_X \in \mathsf{Set}([X], X)$ is usual identity function $X \to X$

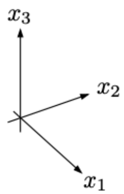
Composition

$$X_1 \times X_2 \times X_3 \xrightarrow{f} Y_1 \qquad X_4 \times X_5 \xrightarrow{g} Y_2 \qquad Y_1 \times Y_2 \xrightarrow{h} Z$$

$$\rightsquigarrow \quad X_1 \times X_2 \times X_3 \times X_4 \times X_5 \xrightarrow{(f \times g);h} Z$$

Swiss cheese operad





$$1, 1, 2, 2, 2, 3 \longrightarrow 1$$